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TRACTS:

CONTAINING

I, MECHANICS, or the DOCTRINE of MOTION.

II. The PROJECTION of the SPHERE.

III. The LAWS of CENTRIPETAL and CENTRIFUGAL FORCE.

By WILLIAM EMERSON.

A NEW EDITION.

TO WHICH IS PREFIXED,

Some ACCOUNT of the LIFE and WRITINGS of the AUTHOR,

By the Rev. W. BOWE,

Of *Scorton*, near *Catterick*, *Yorkshire*.

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MDCCXCIII.

SOME ACCOUNT

OF THE

LIFE

OF

MR. WILLIAM EMERSON.

THE Writer of the following Memoirs judges it proper to premiſe from what ſources he has derived his information. He had the good fortune to be perſonally acquainted with Mr. EMERSON during the laſt three or four years of his life; and enjoyed frequent opportunities of learning, from his own mouth, accounts of circumſtances which had taken place at former periods of his life. He has alſo had frequent converſations reſpecting him with ſeveral perſons of veracity, who knew him many years before. He has not, however, accumulated an indiſcriminate maſs of anecdotes, as they were offered to him, but has rejected ſuch as did not appear to him to be authentic, and adopted ſuch only as upon mature examination and enquiry, he had reaſon to think true. In ſhort, he has endeavoured to form and exhibit as juſt a picture of Mr. EMERSON's Life as he could, under ſome peculiar difficulties; and his motive for publiſhing it was a deſire to gratify, in ſome meaſure, that curioſity, which mankind muſt always entertain, to be informed of whatever has relation to thoſe men, who have drawn the

attention of their fellow-creatures by the ſuperior ſplendor of their talents, and who have enlarged the boundaries of human knowledge.*

Mr. DUDLEY EMERSON, of *Hurworth*, near *Darlington*, in the County of *Durham*, had two Sons, WILLIAM, the elder, and DUDLEY, who died whilſt he was young. WILLIAM, who afterwards lived to become ſo eminent a Mathematician, was born at *Hurworth*, in the year 1701, and, as appears by the Pariſh Regiſter, was baptized there on the 10th of June in that year.

" In a vacant leaf of an old prayer book, in " which DUDLEY EMERSON, the father of " WILLIAM, had regiſtered his marriage, and " the births of his ſeveral children, it is written, " William Emerſon was born Wedneſday, " May 14, at one o'clock in the morning, and " 40 minutes, and baptized June 10, 1701."

His father DUDLEY who was poſſeſſed of but a ſmall eſtate, at that time taught a ſchool, and ſeems to have thought himſelf of ſome little conſequence in the world, for I have ſeen a paper, written by himſelf, containing what he calls an account of the principal tranſactions or events of his life. Amongſt theſe *memorabilia*, relating, I think, chiefly to his movements from one place to another, I obſerved nothing reſpecting the birth or education of his ſon *William*; which he did not foreſee would be the only circumſtances or events of any importance in his life, that might poſſibly reſcue his name from oblivion.

William

* " Several circumſtances and anecdotes have been intermixed with Mr. BOWE's narrative, or ſubjoined to it as " notes. Theſe were partly taken from other accounts of " Mr. EMERSON, and partly communicated by gentlemen " who had a perſonal acquaintance with him."

William was taught, and principally by his father, reading, writing and arithmetic, and a little Latin, perhaps as far as *Cordery*, or *Beza's Latin Teſtament*.* It does not appear, however, that he was much attached to his books whilſt a boy, or exhibited any ſymptoms of thoſe ſuperior faculties, which he afterwards exerted with ſo much energy. Indeed ſo careleſs and inattentive to learning was he, at this period, that I have heard him ſay, till he was near twenty years of age, his principal and favourite employment, for one ſeaſon of the year, was that of ſeeking birds neſts. But his attachment to childiſh amuſements was now to paſs away; and his mind began to be ſenſible of the charms and beauties of ſcience. He went firſt to Newcaſtle, and afterwards to York, where he applied himſelf, with conſiderable attention and diligence, to the ſtudy of the Mathematics, under the direction of ſchoolmaſters, whoſe names I do not remember, but of whom he uſed to ſpeak, in the latter part of his life, with much reſpect. "He "uſed to ſay too, that his father was a tolerable "Mathematician; and, without his books and "inſtructions, perhaps his own genius (moſt "eminently fitted for mathematical diſquiſitions) "would never have been unfolded."

After his return from the ſchool at York, he reſided principally at Hurworth, where he continued to purſue his ſtudies and amuſements, at intervals, until the time of his marriage. In what year of his life this happened I do not exactly know, but I think it was about the thirty-ſecond or thirty-third: and from this

* "He afterwards received ſome aſſiſtance in the learned "languages from a young gentleman, then curate of "Hurworth, who was boarded at his father's houſe."

period we muſt date the commencement of his mathematical *labors*; or perhaps rather the communication of them to the public. What he had done before in this line, was merely an occaſional application, for his own amuſement, or for the exerciſe and improvement of his leiſure hours. But one of thoſe accidents, which, as Dr. Johnſon obſerves in the Life of Cowley, "produce that particular deſignation of mind and propenſity for ſome certain ſcience, commonly called genius," took place upon this occaſion, and added a powerful ſtimulus to his native thirſt for knowledge, and for fame. His wife was the Niece of a Doctor Johnſon, Rector of Hurworth, Vicar of Manfield in the county of York, and a Prebendary of Durham, a man very eminent in his time for his ſkill in ſurgery, and who, by a very extenſive and ſucceſsful practice in this profeſſion, together with the emoluments ariſing from his livings, had accumulated a conſiderable fortune. Dr. Johnſon had promiſed to give his niece, who lived with him, five hundred pounds for her marriage portion. Some time after the marriage, Mr. EMERSON took an opportunity to mention this matter to the Doctor, and to remind him of his promiſe. The Doctor, however, did not recollect, or did not chuſe to recollect, any thing of it, but treated our young Mathematician with ſome contempt, as a perſon of no conſequence, and beneath his notice. The pecuniary diſappointment, EMERSON (who had as independent a ſpirit as any man, and whoſe patrimony, though not large, was equal to all his wants) would eaſily have ſurmounted, but this contemptuous treatment ſtung him to the very ſoul. He immediately went home, packed up all his wife's cloaths,

and

and ſent them off to the Doctor, ſaying that he would ſcorn to be beholden to ſuch a fellow for a ſingle rag; and ſwearing at the ſame time that he would be revenged, and prove himſelf to be the better man of the two, His plan of revenge was truly noble and laudable.—He was reſolved to demonſtrate to his uncourteous uncle and to the world, that he was not to be rated as an inſignificant or ignorant perſon; and that the contempt and indignity, with which he had been treated, were much miſplaced, and very unmerited: and, in order to demonſtrate this, he determined to labour till he became one of the firſt Mathematicians of the age.

He had received from nature a ſtrong and vigorous mind, and had acquired a juſt reliſh for the beauties of Mathematical Science, and an ardent love of Truth; he was at the ſame time, ſtimulated with an eager deſire of diſtinguiſhing himſelf from the illiterate crowd of mortals: the effects of his labor, influenced by ſuch motives, and directed by ſuch abilities, could not therefore but be great. He made himſelf a perfect maſter of the whole circle of the Mathematics; and after having carefully planned and digeſted, reviſed and compleated the work to his own ſatisfaction, he publiſhed, in the forty-ſecond year of his age, his book of *Fluxions*; and at his firſt appearance in the world as an author, ſtepped forth like a giant in all his might, and juſtly claimed a place amongſt Mathematicians of the very firſt rank. By the ſtrictly ſcientific manner in which he eſtabliſhed the principles, and demonſtrated the truth of the method of Fluxions in this work, he added another firm and durable ſupport to the noble edifice of the Newtonian Philoſophy, which, by ſome leſs accurate and penetrating obſervers, was

ſuppoſed to have received a violent and dangerous concuſſion from the metaphyſical artillery of the *analyſt*, and the cavils and objections which had been advanced againſt the truth of the fluxionary method.

Having thus ſecured his mathematical fame upon a firm and ſolid Baſis, he continued from time to time to favour and inſtruct the publick with other moſt valuable publications upon the ſeveral branches of the Mathematics. Theſe appeared in the order in which they ſtand arranged below: I have added to each the date of the Author's life.*

Years.	Ætat. An.	
1743 .	42	Fluxions, 8vo.
1749 .	48	Projections of the Sphere, and Elements of Trigonometry, 8vo.
1754 .	53	Mechanics, 4to.
1755 .	54	Navigation, 12mo.
1763 .	62	Arithmetic, Geometry, 8vo. Method of Increments, 4to
1764 .	63	Algebra, 8vo.
1767 .	66	Arithmetic of Infinites and Conic Sections, 8vo.
1768 .	67	Elements of Optics and Perſpective, 8vo.
1769 .	68	Aſtronomy, Mechanics, Centripetal and Centrifugal Forces, 8vo.
1770 .	69	Mathematical Principles of Geography, Navigation and Dialling, Comment on the Principia, with the Defence of Newton. Tracts, 8vo.
1776 .	75	Miſcellanies, 8vo. which was his laſt work.

The above Works, many of them allowed to be the beſt extent upon the ſubjects of which they treat, will remain a laſting monument of Mr. EMERSON's genius, penetration and induſtry to

* New editions of ſeveral of the above works have ſince been publiſhed, with improvements, communicated from the Author, by F. WINGRAVE, Succeſſor to Mr. NOURSE, Bookſeller in the Strand.

to the lateſt times; and render any further eulogium of their Author, as a Man of Science, totally unneceſſary.

" His firſt publications however did not meet with immediate encouragement; ſo that it is probable the reſt would never have appeared, or, at leaſt not in the Author's life time, had he not about the year 1763, been recommended, by his great admirer and friend the late EDWARD MONTAGU, Eſqr. to Mr. JOHN NOURSE, Bookſeller in London, who was himſelf an eminent Mathematician, and well ſkilled in the Newtonian Philoſophy, having had an univerſity education, and been an early aſſociate with the learned Doctors Pemberton and Wilſon, the one the companion of Newton, and Editor of the beſt edition of the Principia, the other of Mr. Robins's Mathematical Tracts. Mr. NOURSE was ſo highly ſenſible of Mr. EMERSON's ſuperior abilities, that he engaged him on very liberal terms, to furniſh a regular courſe of the Mathematics for the uſe of young ſtudents. Mr. EMERSON made a journey to London in the ſummer of the year 1763, to ſettle and fulfil this agreement. Even in London he could not be idle: beſides correcting his ſheets for the preſs,* he took lodgings at a Watchmaker's, near Smithfield, that he might improve himſelf, in that branch of knowledge during his ſtay there."

" Beſides the above regular works, publiſhed in Mr. EMERSON's own name, he wrote ſeveral other fugitive pieces, in the Ladies Diaries and other periodical and miſcellaneous works. In the Ladies Diaries he propoſed and anſwered ſeveral new queſtions under the ſignature *Merones*, an anagram

* " He always made the reviſal himſelf; to truſt no eyes " but his own, being a favourite maxim with him."

anagram of his own name, containing all the letters of it tranſpoſed; the queſtions reſolved by him were as follows, viz. prize, 1736; queſt. 193, 195, 197; prize, 1737; queſt. 205, 206, 207, 209, 210, 215, 217, 221, 223; prize, 1741; queſt. 226, 229; prize, 1742; queſt, 238, 240: and he propoſed the following new queſtions; No. 193, 206, and 220*. Mr. EMERSON alſo took a part in the *Miſcellanea Curioſa Mathematica*, a work publiſhed in quarterly numbers, by Mr. FRANCIS HOLLIDAY, (his friend and correſpondent) from the year 1745 till 1755, in 4to. In this work he reſolved many queſtions, as before in the Diaries, ſometimes under the ſignature of *Merones*, and ſometimes under the ſtill more whimſical one of *Philoſluentimechanalgegeomaſtrolongo*; and probably under ſeveral others." We ſhall now take a view of Mr. EMERSON in his private life, as a man, and a member of ſociety.

Mr. EMERSON was, in perſon, ſomething below the common ſize, but firm, compact, well-made, very active and ſtrong. He had a good, open, expreſſive countenance, with a ruddy complexion, a keen and penetrating eye, and an ardor and eagerneſs of look that was very expreſſive of the texture of his mind. His dreſs was very ſimple and plain, or, what, by the generality of people, perhaps would have been called groteſque and ſhabby. A very few hats ſerved him through the whole courſe of his life, and when he purchaſed one (or indeed any other article of dreſs) it was a matter of perfect indifference

* "See the *Diarian Miſcellany*; which is a republication "of all the uſeful parts of the Ladies Diaries, by Dr. "HUTTON In the Diaries, &c. Mr. EMERSON had alſo "ſome warm controverſies with Wadſon (Mr. Dawſon of "Sedbergh) and other eminent Mathematicians."

ference to him, whether the form and faſhion of it was that of the day, or of half a century before. One of theſe hats, of immenſe ſuperficies, had in length of time loſt its elaſticity, and the brim of it began to droop in ſuch a manner as to prevent his being able to view the objects before him in a direct line. This was not to be endured by an optician: he therefore took a pair of ſheers, and cut it round cloſe by the body of the hat, leaving a little to the front, which he dexterouſly rounded into the reſemblance of the nib of a jockey's cap. His wigs were made of brown or a dirty flaxen coloured hair, which at firſt appeared buſhy and tortuous behind, but which grew pendulous through age, till at length it became quite ſtraight, having probably never undergone the operation of the comb: and either through the original mal-conformation of the wig, or from a cuſtom he had of frequently inſerting his hand beneath it, his hind-head and wig never came in very cloſe contact. His coat, or more properly jacket, or waiſtcoat with ſleeves to it, which he commonly wore without any other waiſtcoat, was of a drab colour. His linen came not from *Holland* or *Hibernia*, but was ſpun and bleached by his wife, and woven at *Hurſworth*, being calculated more for warmth and duration than for ſhew. He had a ſingular cuſtom of frequently wearing, eſpecially in cold weather, his ſhirt with the wrong ſide before, and buttoned behind the neck. But this was not an *affectation* of ſingularity (for EMERSON had no *affectation* about him, though his cuſtoms and manners were ſingular:) he had a *reaſon* for it;—he ſeldom buttoned more than two or three of the buttons of his waiſtcoat, one or two at the bottom, and ſometimes one at the top; leaving all the reſt

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open. In wind, rain or ſnow, therefore, he muſt have found the aperture at the breaſt inconvenient, if his ſhirt had been put on in the uſual manner. His breeches had an antique appearance, the lappet before not being ſupported by two buttons, placed in a line *parallel* to the plane of the horizon, but by buttons deſcending in a line *perpendicular* to it. In cold weather he uſed to wear, when he grew old, what he called *ſhin-covers*. Now theſe *ſhin-covers* were made of old ſacking, tied with a ſtring above the knee, and depending before the ſhins down to the ſhoe; they were uſeful in preſerving his legs from being burnt, when he ſate too near the fire (which old people are apt to do;) and if they had their *uſe*, he was not ſolicitous about the figure or appearance they might make.

This ſingularity of dreſs and figure, together with his character for profound learning, and knowledge more than human, cauſed him to be conſidered, by ignorant and illiterate people in the neighbourhood, as a *wiſe* or *cunning man*, or *conjuror*: many of them are ſtill perſuaded that he was ſuch, and will tell you wonderful ſtories of the feats he performed, and particularly how by virtue of a magic ſpell, he pinned a fellow in the top of his pear or cherry tree, who had got up with a deſign to ſteal his fruit; and compelled him to ſit there a whole Sunday's forenoon, in full view of the congregation going to and returning from church. That he did compel a man to ſit for ſome time in the tree, I believe was a fact, not however by virtue of any magic ſpell, but by ſtanding at the bottom of the tree, with a hatchet in his hand, and ſwearing that if he came down he would *hag* (i.e. hew) *his legs off*. This opinion of his ſkill in the *Black-art*, was of ſervice in defending his property from ſuch depredations, and therefore

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it would have been impolitic to diſcourage it: but he was apt to loſe his patience very much, when he was applied to for the recovery of ſtolen goods, or to inveſtigate the ſecrets of futurity. A woman came one day to him to enquire about her huſband, who had gone ſix years before to the Weſt Indies or America, and had not been heard of ſince. She requeſted therefore to be informed whether he was dead or living, as a man, in her neighbourhood, had made propoſals of marriage to her. It was with much difficulty the ſuppoſed *prophet* repreſſed the riſing *furor* till the concluſion of the tale; when riſing haſtily from the tripod, or three footed ſtool, on which he uſually ſate, in terms more energetic than ever iſſued from the ſhrine at Delphi; he gave this plain and unequivocal reſponſe; "D—n thee, for a bitch! thy huſband's gone to hell; and thou may go after him." The woman went away well pleaſed and ſatisfied with the anſwer ſhe had received, thinking ſhe might now liſten to the propoſals of her lover with a ſafe conſcience. Another young damſel, on a ſimilar errand, met with a milder reception. Her miſtreſs had loſt ſome caps, or linen, and ſhe wanted to know whether her fellow ſervant (of whom ſhe entertained ſuſpicions) had purloined them or not. "*Thou's a canny young laſs*, replied the ſmiling conjurer, *but thou's over late o'coming, I can do nought for the.*" The poor girl went away grieved that ſhe had not made her application ſooner, ſuppoſing he meant that the myſterious moment of divination was paſt. He was by many people alſo looked upon as an atheiſt, but he was as much an atheiſt as he was a magician. He firmly believed in the being of a God; he did not *believe* it, as he ſometimes ſaid, he *knew* it, he was *certain* of it to a demonſtration. But it muſt

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be acknowledged, that he did not always ſpeak of revealed religion, the Church of England or the clergy, in terms of reſpect. It has often been obſerved and lamented, that minds merely mathematical, are apt to tend towards ſcepticiſm and irreligion. The man who is always accuſtomed to demonſtrative proofs, and wholly engaged in a ſcience which admits of them at every ſtep, will not ſo readily acquieſce in a ſeries of probabilities, where inveſtigations of another kind are preſented to him; and perhaps will not have patience to examine circumſtances deeply enough to aſcertain, on which ſide there is a preponderancy of evidence amounting nearly to demonſtration. Beſides, EMERSON's reſentment at Dr. Johnſon's treatment of him, which I have mentioned before, might produce a bias on his mind unfavourable to religion. A man of his natural impetuoſity of temper would be too apt to aſſociate the idea of the *profeſſion* with that of the *profeſſor*; and becauſe he had quarrelled with the *prieſt*, would alſo quarrel with his *doctrine*. Under the influence of this prepoſſeſſion, he ſet himſelf to work to examine the Scriptures of the Old and New Teſtament, and collected two ſmall 4to. vols. of what he conceived to be contradictory paſſages; and arranged them, like hoſtile troops confronting each other, on the oppoſite pages of his book.

His diet was as ſimple and plain as his dreſs, and his meals gave little interruption to his ſtudies, employments or amuſements. During his days of cloſe application, he ſeldom ſat down to eat, but would take a piece of cold pye or meat of any kind in his hand, and retiring with it to his place of ſtudy, could ſatisfy his appetite for knowledge and food at the ſame time. He catered for himſelf, and pretty conſtantly made

made his own market. When his ſtock of groceries, or other neceſſaries in the article of houſekeeping, grew low, on the Monday morning he took his *wallet*, which he ſlung obliquely acroſs his ſhoulders, and ſet forward for the market at Darlington, three miles diſtant from Hurworth; whither he always walked on foot, for he ſeldom or never kept a horſe, and had an averſion to riding. "He would frequently lead "the horſe, when he had one, from market, by "the halter, bearing the wallet ſtuffed with the "proviſions he had bought at market." After having provided all the neceſſary articles he did not always make directly home again, but, if he found good fair ale, and company to his mind, he would ſit himſelf down contentedly in ſome public houſe, for the remainder of the day, and frequently during the night too; ſome times he did not reach home till late on Tueſday or even Wedneſday; he remained talking, or diſputing on various topics, mechanics, politics or religion, juſt as his company might be, varying the ſcene ſometimes with a beef-ſtake, mutton-chop, or a pan of cockles; for it is remarkable that his ale did not injure, but rather improve his appetite; and that he never felt the head-ach, or any immediate ill effects afterwards. In theſe *durable* potations, he would ſometimes indulge, not only at Darlington, but in Hurworth, or ſome neighbouring village, and always in an ale-houſe, for he kept no ſtock of ale at home; and he was upon all occaſions rigidly exact in apportioning each man's quota of the reckoning. The laſt time he made an excurſion to Darlington with his wallet, our Philoſopher made a figure truly conſpicuous: this was the only time, I believe, he ever rode thither,

and

and he was then mounted on a quadruped, whoſe intrinſic value, independent of the ſkin, might be fairly eſtimated at half a crown. Being preceded and led by a boy hired for that purpoſe, he crawled in ſlow and ſolemn ſtate, at the rate of a mile and a half in an hour, till in due time he arrived at Darlington, and was conducted in the ſame ſtate, to the great entertainment of the ſpectators through the ſtreets to the inn, where he wiſhed to refreſh himſelf and his beaſt. What idea EMERSON himſelf entertained of the velocity with which the animal could move, appears from this, that when a neighbour of his from Hurworth aſked him towards the evening, if he was going home? *D—n thee* (ſays he) *what does thou want with my going home?* Only, ſays the man, becauſe I ſhould be glad of your company.—*Thou fool, thou!* (rejoined the other) *Thou'lt be home long enough before me, man! thou walks and I ride.* We muſt obſerve, that ſuch expreſſions as *D—n thee*, and *thou fool*, were mere expletives often with him, expreſſive neither of indignation nor contempt.

His ſtyle in converſation was generally very abrupt and blunt, abounding in ſuch expletives as the above, and often vulgar and ungrammatical. And this was the reaſon that led many people to ſuppoſe, he could not be able to write any thing like grammar, or tolerable Engliſh, and that his prefaces therefore, ſome of which are not ill written, could not be his own compoſition, but muſt have been made, or at leaſt tranſlated into Engliſh for him by ſome other hand. But we cannot, with certainty, judge how a man will write from his language in common converſation. Though EMERSON was no accurate grammarian, his reading was extenſive enough to ſupply him with a ſufficient

a ſufficient ſtock of proper words and expreſſions on ſuch ſubjects as he had occaſion to treat of, and the vigour and energy of his mind would give force, weight and perſpicuity to his ſentences. He was one day told, that this opinion reſpecting his prefaces was entertained by many; and the diſparity of his converſation and writing was pointed out as the reaſon of it. After a momentary pauſe he exclaimed, with ſome indignation: *A pack of fools! who could write my prefaces but my ſelf?* They do in fact carry with them every mark of legitimacy; they could have no other father. Indeed the original prefaces, as well as the original MSS. of moſt of the Author's works, in his own hand writing, with a great number of his original letters, are now in the poſſeſſion of the publiſher, all of which are well written, and in a good ſtile. There are alſo, in ſome of them, quotations from the Greek Authors, elegantly written by himſelf. In the earlier part of his life, principally by the aid of his Dictionary, he had acquired a ſtock of latin, ſufficient to enable him to read and tranſlate mathematical works in that language: and he once had a deſign to give a tranſlation of the Jeſuits comment, on the Principia. It appears alſo from his *mottos* to his ſeveral books, that he ſometimes *dipped* into the Claſſic Authors, for we muſt not ſuſpect him of having been *much converſant* with ſuch gentry as *Lucretius*, *Horace* and *Virgil*. He had in his library *Homer* and *Virgil traveſtied*, and theſe he read with more pleaſure than *Pope's* or *Dryden's* tranſlations. In arguments or diſputes, whether in converſation or writing, the fire and impetuoſity of his temper was too apt to break forth, and betray him into the uſe of ſuch terms, as ſhould always be avoided, eſpecially by Mathematicians and Philoſophers, whoſe employment

is, or ought to be, the cool and diſpaſſionate ſearch after truth. His zeal in ſupport of Sir Iſaac Newton is outrageous, and he has treated the oppugners of his Philoſophy as if he had a perſonal quarrel againſt them. "*J. Bernoulli* (according to him) is a low critic, laborious and tedious to the laſt degree: a blunderbuſs, a perſon of eternal contradiction, blind, bigotted, prejudiced, mad; and a whole ſection of 100 pages is nothing but a heap of abſurd, inconſiſtent ſtuff." He therefore leaves him (in his great mercy) to be drowned in a gurges of his own contriving. *Euler* receives no milder treatment: he is vortex-mad too, and is one of thoſe bigotted Philoſophers, whoſe brains are turned in a vortex, or they never would prefer ſuch complex vortical ſchemes before the ſimple doctrine of projectile and centripetal forces. Such Philoſophers! exclaims our indignant defender. Leibnitz's attempt to rob Sir Iſaac Newton of the honour of the invention of Fluxions, is, as we might expect, a cauſe ſufficient to make him rave with rage, and to call down all the thunder of his indignation. To tell us flatly "that not he that firſt *found*, but he that firſt *publiſhed*, deſerves the praiſe," is an aſſertion that might move the bile in a ſtomach leſs apt to boil than EMERSON's. It is robbing, as he truly ſays, the inventor of his due praiſe to give it to the thief that ſtole it. But however much Sir Iſaac might have approved of Mr. EMERSON as the *champion* or *defender* of his principles, he would not have admired the *weapons* he ſometimes made uſe of.

In ſo difficult a problem as EMERSON himſelf allows that of finding the preceſſion of the equinoxes to be, one would have thought that he would have had ſome patience with thoſe who differ

differ from Sir Iſaac and himſelf. But theſe are a *clan*, and Mr T. Simpſon has the misfortune to be one of the *clan*. There ſeems, unfortunately, to have ſubſiſted ſome kind of animoſity or jealouſy between theſe two great contemporary Mathematicians, excited and fomented perhaps by ſome, who were mean enough to carry tales between them, in order to curry favor. One of EMERSON's correſpondents, in London, informed him that Simpſon had ſaid, in company with ſome mathematical friends, " that he took EMERSON (referring to his book of Fluxions) to be an induſtrious, plodding writer, but a man of no genius." This was quite enough to irritate EMERSON to acts of hoſtility.

We meet with frequent complaints in his books of the little encouragement given to mathematical learning in his time, and eſpecially in England. How far this complaint may be well or ill founded, I know not, but this is pretty certain, that if any reward or recompenſe had been offered to him for his mathematical labors, and had not come to him in his own way, he would not have accepted it. He did not wiſh to be admitted a fellow of the Royal Society, " *becauſe* (he ſaid) *it was a d—n'd hard thing that a man ſhould burn ſo many farthing candles as he had done, and then have to pay ſo much a year for the honor of F. R. S. after his name. D—n them and their F. R. S. too.*" He reaped one reward of his toils in queſt of ſcience, which was very acceptable and grateful to him; and that was the acquaintance and friendſhip of *Mr. Montagu*, who having an eſtate at Eryholme, near Hurworth, ſometimes viſited that neighbourhood, and ſpent a good part of his time in company and converſation with EMERSON, and at his deceaſe bequeathed to him a legacy of 100l. or guineas.

When Mrs. Montagu's agent called upon him to diſcharge this legacy, EMERSON told him he would much rather have ſeen Mr. Montagu himſelf than his money. And he ſpoke, doubtleſs, from his heart, for he never mentioned Mr. Montagu's name but in terms that ſtrongly expreſſed the ſincerity and ardor of his affection and eſteem for him.

" Mr. Montagu, in his viſits to EMERSON, would often go to him in the fields, when he happened to be at work there, and would accompany him home, but could never perſuade him to get into a carriage. On theſe occaſions he would ſometimes exclaim, " D--n your whim-wham! I had rather walk."

' Inheriting a ſmall patrimonial eſtate, of about 60 or 70 pounds a year, he was as independent and happy as if he had enjoyed ſo many thouſands. He was never known to aſk a favour, or ſeek the acquaintance of a rich man, unleſs he poſſeſſed ſome eminent qualities of the mind.' " Notwithſtanding his imperfect and deſultory courſe of education, he acquired a general knowledge of moſt ſciences. He had even paid attention to medicine, at leaſt ſo far as it has been combined with mathematical principles, according to the plan of Keil, Morgan, &c. He eſteemed Morgan above all others as a phyſician: and held Keil to be the beſt of anatomiſts.

" Mr. EMERSON often tried in practice the effect of his mathematical ſpeculations, by conſtructing a variety of inſtruments, mathematical, mechanical, or muſical, on a ſmall ſcale. He made a ſpinning wheel for his wife, which is repreſented by a drawing in his book of Mechanics. He was well ſkilled in the ſcience of muſic, the theory of ſounds, and the various

" ſcales,

" ſcales, both ancient and modern; but he was
" a very poor performer, though he ſometimes
" obliged his friends in the country by tuning
" and repairing their muſical inſtruments. He
" carried that ſingularity which marked all his
" actions even into this ſcience. He had, if I
" may be allowed the expreſſion, two firſt ſtrings
" to his violin, which he ſaid made the E more
" melodious when they were drawn up to a perfect
" uniſon. His virginal, which is a ſpecies of in-
" ſtrument like the modern ſpinnet, he had cut
" and twiſted into various ſhapes in the keys, by
" adding ſome occaſional half-tones, to regulate
" the preſent ſcales, and to rectify ſome fraction
" or diſcord that will always remain in the tuning.
" This he never could get regulated to his mind,
" and generally concluded in a paſſion by ſaying,
" It is a d—mn'd inſtrument, and a fooliſh thing
" to be vexed with."

" Mr. EMERSON was very fond of angling, and
" while he thus amuſed himſelf, would ſtand up
" to his middle in water for ſeveral hours toge-
" ther.—When he was building a houſe upon the
" ſmall farm which he poſſeſſed by the ſide of
" the Tees, he never heſitated to plunge into the
" water, for the purpoſe of collecting ſtones
" from the bed of the river. He was affected,
" about that time, with ſome ſlight gouty
" ſymptoms, and ſaid, that *wading* was ſervice-
" able to him, becauſe the water ſucked the gout
" out of his legs; a theory for which he was
" probably not indebted to his favourite Keil or
" Morgan. When he wrote his treatiſe on
" Navigation, he muſt needs make and fit up a
" ſmall veſſel; with this he and ſome young
" friends embarked on the river Tees, that ran
" by his door at Hurworth; but the whole crew

" got

" got ſwampt frequently; when EMERSON, " ſmiling and alluding to his book, ſaid, They " muſt not do as I do, but as I ſay."

" In the earlier part of his life he attempted " to teach a few ſcholars; but whether from " his conciſe method, for he was not happy in " explaining his ideas, or the natural warmth of " his temper, he made little progreſs in his ſchool, " which he therefore ſoon dropped. He never " had a ſcholar that did him any credit except " Mr. Richardſon of Darlington, who was always " a great favourite with him, and of whom he " uſed to ſay, that he was the only boy who had " a head in his ſchool."

" Being requeſted once, by letter, to commu- " nicate ſome particulars of his life, by a friend " who wanted to put them together, he wrote for " anſwer, I never knew you were commenced " biographer before; they will have little to do, " I think, that ſet about writing my life: I am " ſure of this, half of it will be lies: therefore " I chuſe to die in the ſame obſcurity I have " lived."

During the greateſt part of his life he had enjoyed ſtrong and uninterrupted health; but as he advanced in years he was afflicted with the ſtone and gravel to an excruciating degree. In the agony of theſe fits he would crawl round the floor, on his hands and knees, ſometimes praying and ſometimes ſwearing, and devoutly wiſhing that the mechaniſm of the human frame had been ſo contrived as to go to wreck without all that *clitter-my-clatter*, as he called it; as he grew weaker the violence of the diſorder abated, and he died at laſt, apparently without much pain, on the 21ſt of May 1782, and in the 81ſt year of his age.

Mr.

" Mr. EMERSON, with much perſuaſion, about " a year before his deceaſe, was prevailed on by " his friend Dr. Cloudeſley, of Darlington, to " ſit for his picture, which was taken by Sikes, " and is now in the Doctor's poſſeſſion. Towards " the cloſe of the year 1781, being ſenſible of " his approaching diſſolution, he diſpoſed of the " whole of his mathematical library to a book- " ſeller at York: his inſtruments he had for " many years been in the habit of diſpoſing of, " at times, to different perſons, for a mere trifle. " When aſked, why he choſe to ſell his books, " he ſaid, he had none but a pack of fools to " leave them to, and money would be of more " ſervice to them than books. He valued his " library when it was ſent to York at 40 or 50*l.*"

EMERSON, like other great men, had his foibles and defects.—He was ſingular and uncouth in his dreſs and manners, and haſty and impetuous in his temper; but whatever failings he had, they were overbalanced by his virtues. He had a great, firm, and independent mind, that could not be brought to ſubmit to any thing mean, baſe or diſingenuous, by any power on earth: a pure, genuine, ardent love of truth, and deteſtation of falſhood of whatever ſpecies. His honeſty and intregrity were ſuch, that all who knew any thing of him, repoſed in him the moſt implicit confidence; and no man could ever juſtly complain that EMERSON had deceived him. He had great pleaſure in doing a good and friendly ſervice to any deſerving perſon, whenever he had it in his power; and under a rough and forbidding exterior he concealed a humane heart, that wiſhed to promote the welfare and happineſs of his fellow creatures.

He lies buried in the church-yard at Hurworth, at the weſt end of the church; againſt which is erected to his memory a ſtone with the following inſcription. His wife ſurvived him near two years; they had no children.

Quod ſub Pedibus ſepultum
Et neglectum jacet
Aliquando fuit
GULIELMUS EMERSON;
Vir
Priſcæ Simplicitatis,
Summæ integritatis,
Rariſſimi ingenii.
Quantus fuerit Mathematicus
Si ſcripta ejus perlegeris
Quorſum narraret ſaxum?
Si non perlegeris,
Perlege, et Scies.
Obiit 21º Mäij, 1782.
Ætat. Anº 81.

Juxta ſepulta jacet
ELIZABETHA UXOR.
Quæ obiit 27º Martii, 1784,
Ætat. An. 76.

MECHANICS;

OR, THE

DOCTRINE

OF

MOTION.

COMPREHENDING,

I. The general LAWS of MOTION.

II. The DESCENT of BODIES perpendicularly, and down inclined PLANES, and alſo in CURVE SURFACES. The MOTION of PENDULUMS.

III. CENTERS of GRAVITY. The EQUILIBRIUM of BEAMS of TIMBER, and their FORCES and DIRECTIONS.

IV. The MECHANICAL POWERS.

V. The COMPARATIVE STRENGTH of TIMBER, and its STRESS. The POWERS of ENGINES, their MOTION and FRICTION.

VI. HYDROSTATICS and PNEUMATICS.

Da veniam ſcriptis, quorum non gloria nobis
cauſa, ſed utilitas officiumque, fuit.

OVID DE PONTO III.

PREFACE.

HAVING some years since written a large book of Mechanics, I have here given a short abstract of that work, as it falls properly into this course; and especially as that branch of science, is of such extensive use in the affairs of human life. I do not in the least, design this to interfere with the other book, being rather an introduction to it, as it explains several things in it more at large; particularly in the first section, as being of universal extent and use; and likewise in several other parts of the book, especially such as have been objected to by ignorant writers. I have also added several things not mentioned in the other book, which are more simple and easy, and more proper for learners. So that this short treatise may be looked upon as an introduction to the other book, and will doubtless facilitate the reading of it. As to the higher and more difficult matters, as few care to trouble their heads about them, I have said little of them here, being not so proper for an introduction. To mention one or two things; I had taken a great deal of pains to find out the true form of a bridge, that shall be the strongest, and of a ship that shall sail the fastest; both upon principles that I know to be as certain and demonstrative as the Elements of Euclid; both these you have in the other book. But, as we have no occasion in England, *for the strongest bridges, or the swiftest ships; Mathematicians, for the future, may find something else to do, than run into such perplext and useless disquisitions. For indeed when any*

of theſe grand things are to be performed, they generally fall into the hands of ſuch people, as know little of the nature of them. They perhaps know how to lay down upon paper, the plan of a deſign, by rule and compaſs, and to do ſeveral problems in practical geometry; and not much more. As if this was an adequate qualification, for conducting ſuch magnificent works, as coſt many thouſand pounds to execute. So that theſe things, inſtead of being conſtructed by the rules of art; they are too often done by fancy, without any true rules.

But in this there is no great wonder, conſidering how few people ſtudy this art; and among thoſe that do, how few are competent judges. For even among thoſe that preſume to write about it, it is ſurprizing to ſee what miſtakes they daily run into. One denies the compoſition aud reſolution of forces; another cannot be ſatisfied with ſome of the mechanical powers, not even ſo ſimple a thing as the wedge; all owing to the wrong notions they have imbibed. And in conſequence of this, will be either condemning the art itſelf, or criticiſing on other writers, whom they do not underſtand.

As to what I have written on this ſubject; I have all along given the demonſtrations of the ſeveral things I have handled; and I expect that *to be my teſt, as to the truth or falſhood thereof. And by this teſt I leave them to be tried by any judicious, honeſt reader; who is a lover of truth, and a promoter of ſcience.*

W. Emerſon.

MECHANICS.

DEFINITIONS.

1. *MECHANICS* is a ſcience, which treats of the forces, motions, velocities, and in general, of the actions of bodies upon one another. It teaches how to move any given weight with any given power; how to contrive engines to raiſe great weights, or to perform any kind of motion.

2. *Body* is the maſs or quantity of matter; an *elaſtic body* is that which yields to a ſtroke, and recovers its figure again. But if not, it is called an *unelaſtic body*.

3. *Denſity* is the proportion of the quantity of matter in any body, to the quantity of matter in another body of the ſame bigneſs.

4. *Force* is a power exerted on a body to move it. If it act inſtantaneouſly, it is called *percuſſion*, or *impulſe*. If conſtantly, it is an *accelerative force*.

5. *Velocity* is a property of motion, by which a body paſſes over a certain ſpace in a certain time. And is greater or leſſer, as it paſſes over a greater or leſſer ſpace in a certain time, as ſuppoſe a ſecond.

6. *Motion* is a continual and ſucceſſive change of place. If the body moves equally, it is called *equable* or *uniform motion*. If it increaſes, or decreaſes it is called *accelerated* or *retarded motion*. When it is compared with ſome body at reſt, it is called

abſolute motion. But when compared with others in motion, it is called *relative motion.*

7. *Direction* of motion is the courſe or way the body tends, or the line it moves in.

8. *Quantity* of motion, is the motion a body has, conſidered both in regard to its velocity and quantity of matter. This is alſo called the *Momentum* of a body.

9. *Vis inertiæ*, is the innate force of matter, by which it reſiſts any change, ſtriving to preſerve its preſent ſtate of reſt or motion.

10. *Gravity* is that force wherewith a body endeavours to fall downwards. It is called *abſolute gravity* in empty ſpace; and *relative gravity* when immerſed in a fluid.

11. *Specific gravity*, is the greater or leſſer weight of bodies of the ſame magnitude, or the proportion between their weights. This proceeds from the natural denſity of bodies.

12. *Center of gravity*, is a certain point of a body; upon which, the body, when ſuſpended, will reſt in any poſition.

13. *Center of motion*, is a fixed point about which a body moves. And the *axis of motion* is a fixed line it moves about.

14. *Power* and *weight*, when oppoſed to one another, ſignify the body that moves another, and the other which is moved. The body which begins and communicates motion is the *power*; and that which receives the motion, is the *weight.*

15. *Equilibrium* is the balance of two or more forces, ſo as to remain at reſt.

16. *Machine* or *engine*, is any inſtrument to move bodies, made of levers, wheels, pullies, &c.

17. *Mechanic powers*, are the balance, lever, wheel, pulley, ſcrew and wedge.

18. *Streſs* is the effect any force has to break a beam, or any other body; and *ſtrength* is the reſiſtance

reſiſtance it is able to make againſt any ſtraining force.

19. *Friction* is the reſiſtance which a machine ſuffers, by the parts rubbing againſt one another.

POSTULATA.

1. That a ſmall part of the ſurface of the earth may be looked upon as a plane. For though the earth be round, yet ſuch a ſmall part of it as we have any occaſion to conſider, does not ſenſibly differ from a plane.

2. That heavy bodies deſcend in lines parallel to one another. For though they all tend to a point which is the center of the earth, yet that center is at ſuch a diſtance that theſe lines differ inſenſibly from parallel lines.

3. The ſame body is of the ſame weight in all places on or near the earth's ſurface. For the difference is not ſenſible in the ſeveral places we can go to.

4. Though all matter is rough, and all engines imperfect; yet, for the eaſe of calculation, we muſt ſuppoſe all planes perfectly even; all bodies perfectly ſmooth; and all bodies and machines to move without friction or reſiſtance; all lines ſtraight and inflexible, without weight or thickneſs; cords extremely pliable, and ſo on.

AXIOMS.

1. Every body endeavours to remain in its preſent ſtate, whether it be at reſt, or moving uniformly in a right line.

2. The alteration of motion by any external force is always proportional to that force, and in direction of the right line in which the force acts.

3. Action and re-action, between any two bodies, are equal and contrary.

4. The

4. The motion of any body is made up of the ſum of the motions of all the parts.

5. The weights of all bodies in the ſame place, are proportional to the quantities of matter they contain, without any regard to their figure.

6. The *vis inertiæ* of any body, is proportional to the quantity of matter.

7. Every body will deſcend to the loweſt place it can get to.

8. Whatever ſuſtains a heavy body, bears all the weight of it.

9. Two equal forces acting againſt one another, in contrary directions, deſtroy one anothers effects. And unequal forces act only with the difference of them.

10. When a body is kept in equilibrio; the contrary forces in any line of direction are equal.

11. If a certain force generate any motion; an equal force acting in a contrary direction, will deſtroy as much motion in the ſame time.

12. If a body be acted on by any power in a given direction; it is all one in what point of that line of direction, the power is applied.

13. If a body is drawn by a rope, all the parts of the rope are equally ſtretched. And the force in any part acts in direction of that part. And it is the ſame thing whether the rope is drawn out at length, or goes over ſeveral pullies.

14. If ſeveral forces at one end of a lever, act againſt ſeveral forces at the other end; the lever acts and is acted on in direction of its length.

SECT. I.

The General Laws of MOTION.

PROP. I.

THE quantities of matter in all bodies, are in the compound ratio of their magnitudes and denſities.

For (Def. 3.) in bodies of the ſame magnitude, the quantities of matter will be as the denſities. Increaſe the magnitude in any ratio, and the quantity of matter is increaſed in the ſame ratio. Conſequently the quantity of matter is in the compound ratio of the denſity and magnitude.

Cor. 1. *In two ſimilar bodies, the quantities of matter are as the denſities, aud cubes of the diameters.*

For the magnitudes of bodies are as the cubes of the diameters.

Cor. 2. *The quantities of matter are as the magnitudes and ſpecific gravities.*

For (by Def 3. and 11.) the denſities of bodies are as their ſpecific gravities.

PROP. II.

The quantities of motion in all moving bodies, are in the compound ratio of the quantities of matter and the velocities.

For if the velocities be equal, the quantities of motion will manifeſtly be as the quantities of matter. Increaſe the velocity in any ratio, and the quantity of motion will be increaſed in the ſame ratio.

ratio. Therefore it follows univerſally, that the quantities of motion are in the compound ratio of the velocities and quantities of matter.

Cor. *Hence if the body be the ſame, the motion is as the velocity. And if the velocity be the ſame, the motion is as the body or quantity of matter.*

PROP. III.

In all bodies moving uniformly, the ſpaces deſcribed, are in the compound ratio of the velocities and the times of their deſcription.

For in any moving body, the greater the velocity, the greater is the ſpace deſcribed; that is, the ſpace will be as the velocity. And in twice or thrice the time, &c. the ſpace will be twice or thrice as great; that is, the ſpace will be increaſed in proportion to the time. Therefore univerſally the ſpace is in the compound ratio of the velocity, and the time of deſcription.

Cor. 1. *The time of deſcribing any ſpace, is as the ſpace directly and velocity reciprocally; or as the ſpace divided by the velocity. And if the velocity be the ſame, the time is as the ſpace. And if the ſpace be the ſame, the time is reciprocally as the velocity.*

Cor. 2. *The velocity of a moving body, is as the ſpace directly, and time reciprocally; or as the ſpace divided by the time. And if the time be the ſame, the velocity is as the ſpace deſcribed. And if the ſpace be the ſame, the velocity is reciprocally as the time of deſcription.*

PROP: IV.

The motion generated by any momentary force, or by a ſingle impulſe, is as the force that generates it.

For if any force generates any quantity of motion; double the force will produce double the motion; and treble the force, treble the motion, and ſo on. If a body ſtriking another, gives it any motion, twice that body ſtriking the ſame, with the ſame velocity, will give it twice the motion, and ſo the motion generated in the other will be as the force of percuſſion.

Cor 1. *Hence the forces are in the compound ratio of the velocities and quantities of matter.*

For (Prop. II.) the motions are as the quantities of matter multiplied by the velocities.

Cor. 2. *The velocity generated, is as the force directly and quantity of matter reciprocally. Therefore if the bodies are equal, the velocities are as the forces. And if the forces are equal, the velocities are reciprocally as the bodies.*

Cor. 3. *The quantity of matter, is as the force directly, and velocity reciprocally. And therefore if the velocities be equal, the bodies are as the forces. And if the forces be equal, the bodies are reciprocally as the velocities.*

PROP. V.

The quantity of motion generated, by a conſtant and uniform force, is in the compound ratio of the force and time of acting.

For the motion generated in any given time will be proportional to the force that generates it; and

and in twice that time, the motion will be double by the ſame force; and in thrice that time it will be treble; and ſo every part of time adds a new quantity of motion equal to the firſt; and therefore the whole motion, will be as the force, and the whole time of acting.

Cor. 1. *The motion loſt in any time, is in the compound ratio of the force and time.*

Cor. 2. *The velocity generated (or deſtroyed) in any time, is as the force and time directly, and quantity of matter reciprocally. The ſame is true of the increaſe or decreaſe of velocity.*

For the motion, that is (Prop. II.) the body multiplied by the velocity, is as the force and time. And therefore the velocity is as the force and time directly, and the body reciprocally.

Cor. 3. *Hence if the force be as the quantity of matter, the velocity is as the time. Or if the force and quantity of matter be given, the velocity is as the time.*

And if the time and quantity of matter be given, the velocity is as the force.

And if the force and time be given, the velocity is reciprocally as the matter.

Cor. 4. *The time is as the quantity of matter and velocity directly, and the force reciprocally. Therefore,*

If the force and velocity be given, the time is as the quantity of matter.

If the quantity of matter and velocity be given, the time is reciprocally as the force.

Cor. 5. *The force is as the quantity of matter and velocity directly, and the time reciprocally. Whence,*

If the velocity is as the time, or if the velocity and time be given, then the force is as the quantity of matter.

And

And if the velocity and quantity of matter be given, the force is reciprocally as the time.

Cor. 6. *The quantity of matter is as the force and time directly, and the velocity reciprocally. Therefore, if the force and time be given, the quantity of matter is reciprocally as the velocity.*

Cor. 7. *Hence also if the body be given, the velocity is in the compound ratio of the force and time.*

And if the force be given, the time is in the compound ratio of the matter and velocity, or as the quantity of motion.

PROP. VI.

If a given body is urged by a constant and uniform force; the space described by the body from the beginning of the motion, will be as the force and square of the time.

Suppose the time divided into an infinite number of equal parts or moments. Then in each of these moments of time, the space described (Prop. III. Cor. 2.) will be as the velocity gained; that is, (by Cor. 7. Prop. V.) as the force and time from the beginning. And the sum of all the spaces, or the whole space described, will be as the force and the sum of all the moments of time from the beginning. Therefore put $t =$ the whole time, and the whole space described, will be as the force and sum of the times 1, 2, 3, 4, &c. to t. But the sum of the arithmetic progression $1+2+3+4 \ldots$ to $t = \frac{t+1}{2} \times t = \frac{1}{2}tt$, because t is infinite or consists of an infinite number of moments. Therefore the whole space described will be as the force and $\frac{1}{2}tt$; that is, (because $\frac{1}{2}$ is a given quantity), as the force and the square of the time of description.

Cor.

Cor. 1. *If a body is impelled by a conſtant and uniform force; the ſpace deſcribed from the beginning of the motion, is as the velocity gained, and the time of moving.*

For the ſpace is as the force and ſquare of the time, or as the force × time × time. But (Cor. 7. Prop. V.) the force × time is as the velocity; therefore the ſpace, which is as the force × time × time, is as the velocity × time.

Cor. 2. *If a body urged by any conſtant and uniform force, deſcribes any ſpace; it will deſcribe twice that ſpace in the ſame time, by the velocity acquired.*

For the ſum of all the ſpaces deſcribed by that force, $1+2+3$ &c. to t, was ſhewn to be $\frac{1}{2}tt$. But the ſum of all the ſpaces deſcribed by the laſt velocity, will be $t+t+t$ &c. to t terms, whoſe ſum is tt. But tt is double to $\frac{1}{2}tt$; that is, the ſpace deſcribed by the laſt velocity, is double the ſpace deſcribed by the accelerating force.

Cor. 3. *Univerſally, in all bodies urged by any conſtant and uniform forces, the ſpace deſcribed is as the force and ſquare of the time directly, and the quantity of matter reciprocally.*

For (Cor. 1.) the ſpace is as the time and velocity. But (Prop. V. Cor. 2.) the velocity is univerſally as the force and time directly, and quantity of matter reciprocally. Therefore the ſpace is as the ſquare of the time and the force directly, and matter reciprocally; whence,

Cor. 4. *The product of the force and ſquare of the time, is as the product of the body and ſpace deſcribed.*

Cor. 5. *The product of the force and time, is as the product of the quantity of matter and velocity.*

For (Prop. V.) the product of the force and time,

time, is as the motion; that is, as the body and velocity. Fig.

Cor. 6. *The product of the body, and ſquare of the velocity, is as the product of the force and the ſpace deſcribed.*

For (Cor. 5.) the product of the body and velocity, is as the force and time. Therefore, the body × velocity ſquare, is as the force × time × velocity; but time × velocity is as the ſpace (by Prop. III.); therefore body × velocity ſquare is as the force × ſpace.

Scholium.

If any quantity or quantities are given, they muſt be left out. And ſuch quantities as are proportional to each other muſt be left out. For example, if the quantity of matter be always the ſame; then (Cor. 3.) the ſpace deſcribed is as the force and ſquare of the time. And if the matter be proportional to the force, as all bodies are in reſpect to their gravity; then (Cor. 6.) the ſpace deſcribed is as the ſquare of the velocity. Or if the ſpace deſcribed be always proportional to the body; then (Cor. 6.) the force is as the ſquare of the velocity. Again, if the body be given, then (Cor. 4.) the ſpace is as the force and ſquare of the time. And if both the quantity of matter and the force be given, the ſpace deſcribed is as the ſquare of the time. And ſo of others.

PROP. VII.

If ABCD *be a parallelogram; and if a body at* A, *be acted upon ſeperately by two forces, in the directions* AB *and* AC, *which would cauſe the body to be carried through tho ſpaces* AB, AC, *in the ſame time. Then both forces acting at once, will cauſe the body to be carried through the diagonal* AD *of the parallellogram.* 1.

Let the line AC be ſuppoſed to move parallel to

Fig. 1. itſelf; whilſt the body at the ſame time moves from A, along the line AC or *bg*, and comes to *d* at the ſame inſtant, that AC comes to *bg*. Then ſince the lines AB, AC, are deſcribed in the ſame time; and A*b*, A*d*, are alſo deſcribed in the ſame time. Therefore, as the motions are uniform, it will be, A*b* : *bd* : : AB : BD; and therefore A*d*D is a ſtraight line, coinciding with the diagonal of the parallelogram.

Cor. 1. *The three forces in the directions* AB, AC, AD, are reſpectively as the lines AB, AC, AD.

Cor. 2. *Any ſingle force* AD *denoted by the diagonal of a parallelogram, is equivalent to two forces denoted by the ſides* AB, AC.

Cor. 3. *And therefore any ſingle force* AD *may be reſolved into two forces, an infinite number of ways, by drawing any two lines* AB, BD, *for their quantities and directions.*

Scholium.

This practice of finding two forces equivalent to one, or dividing one force into two; is called the *compoſition* and *reſolution* of forces.

PROP. VIII.

2. *If three forces,* A, B, C, *keep one another in equilibrio; they will be proportional to three ſides of a triangle, drawn parallel to their ſeveral directions,* DI, CI, CD.

Produce AD to I, and BD to H, and compleat the parallelogram DICH; then (Prop. VII.) the force in direction DC, is equal to the forces DH, DI, in the directions DH, DI. Take away the force DC and putting the forces DH, DI equal thereto;

thereto; and the equilibrium will ſtill remain. Therefore (Ax. 10.) DI is equal to the force A oppoſite to it; and DH or CI equal to its oppoſite force B. And as CD repreſents the force C, the three forces A, B, and C, will be to one another as DI, CI, and CD. Fig. 2.

Cor. 1. *Hence if three forces acting againſt one another, keep each other in equilibrio; theſe forces will be reſpectively as the three ſides of a triangle drawn perpendicular to their lines of direction; or making any given angle with them, on the ſame ſide.*

For this triangle will be ſimilar to a triangle whoſe ſides are parallel to the lines of direction.

Cor. 2. *If three active forces* A, B, C, *keep one another in equilibrio; they will be reſpectively as the ſines of the angles, which their lines of direction paſs through.*

For A, B, C are as DI, CI, CD; that is, as S.DCI, S.CDI, and S.DIC. But S.DCI=SCDH =S.CDB. And S.CDI=S.CDA. Alſo S.DIC =S.BDI=SBDA.

Cor. 3. *If ever ſo many forces acting againſt one another, are kept in equilibrio, by theſe actions; they may be all reduced to two equal and oppoſite ones.*

For any two forces may, by compoſition, be reduced to one force acting in the ſame plane. And this laſt force, and any other may likewiſe be reduced to one force acting in the plane of theſe; and ſo on, till they all be reduced at laſt to the action of two equal and oppoſite ones.

PROP. IX.

If a body impinges or acts againſt any plain ſurface; it exerts its force in a line perpendicular to that ſurface. 3.

Let the body A moving in direction AB, with a given velocity, impinge on the ſmooth plain FG at

Fig. 3. the point B. Draw AC parallel, and BC perpendicular to FG; and let AB represent the force of the moving body. The force AB is, by the resolution of forces, equivalent to AC and CB. The force AC is parallel to the plain, and therefore has no effect upon it; and therefore the surface FG is only acted upon by the force CB, in a direction perpendicular to the surface FG.

Cor. 1. *If a body impinges upon another body with a given velocity; the quantity of the stroke is as the sine of the angle of incidence.*

For the absolute force is AB, and the force acting on the surface FG is CB. But AB : CB : : rad. : S.CAB or ABF.

Cor 2. *If an elastic body* A *impinges upon a hard or elastic plane* FG; *the angle of reflection will be equal to the angle of incidence.*

For if AD be parallel to FG; the motion of A in direction AD parallel to the plane, is not at all changed by the stroke. And by the elasticity of one or both bodies, the body A is reflected back to AD in the same time it moved from A to B; let it pass to D; then will AC=CD, being described in equal times; consequently the angle ABC = angle CBD; and therefore the angle DBG = angle ABF.

Cor. 3. *If a non-elastic body strikes another non-elastic body, it loses but half the motion, that it would lose, if the bodies were elastic.*

For non-elastic bodies only stop, without receding from one another; but elastic bodies recede with the same velocity.

Fig.

PROP. X.

The ſum of the motions of two or more bodies, in any direction towards the ſame part, cannot be changed by any action of the bodies upon each other.

Here I reckon progreſſive motions affirmative; and regreſſive ones negative, and to be deducted out of the reſt to get the ſum.

1. If two bodies move the ſame way; ſince action and re-action are equal and contrary, what one body gains the other loſes; and the ſum remains the ſame as before. And the caſe is the ſame, if there were more bodies.

2. If bodies ſtrike one another obliquely; they will act on one another in a line perpendicular to the ſurface acted on. And therefore by the law of action and re-action there is no change made in that direction.

3. And in a direction parallel to the ſtriking ſurface, there is no action of the bodies, therefore the motion remains the ſame in that direction. Whence the motions will remain the ſame in any one line of direction.

Cor. 1. *Motion can neither be increaſed nor decreaſed, conſidered in any one direction; but muſt remain invariably the ſame for ever.*

This follows plainly from this Prop. for what motion is gained to one (by addition) is loſt to another body (by ſubtraction); and ſo the total ſum remains the ſame as before.

SCHOLIUM.

This Prop. does not include or meddle with ſuch motions as are eſtimated in all directions. For upon this ſuppoſition, motion may be increaſed or

Fig. decreaſed an infinite number of ways. For example, if two equal and non-elaſtic bodies, with equal velocities, meet one another; both their motions are deſtroyed by the ſtroke. Here at the beginning of the motion, they had both of them a certain quantity of motion, but to be taken in contrary directions; but after the ſtroke they had none.

PROP. XI.

The motion of bodies included in a given ſpace, is the ſame, whether that ſpace ſtands ſtill; or moves uniformly in a right line.

For if a body be moving in a right line, and any force be equally impreſſed both upon the body and the line it moves in; this will make no alteration in the motion of the body along the right line. And for the ſame reaſon, the motions of any number of bodies in their ſeveral directions will ſtill remain the ſame; and their motions among themſelves will continue the ſame, whether the including ſpace is at reſt, or moves uniformly forward. And ſince the motions of the bodies among themſelves; that is, their relative motions remain the ſame, whether the ſpace including them be at reſt, or has any uniform motion. Therefore their mutual actions upon one another, muſt alſo remain the ſame in both caſes.

SECT.

SECT. II.

The perpendicular deſcent of heavy bodies, their deſcent upon inclined Planes, and in Curve Surfaces. The Motion of Pendulums.

PROP. XII.

THE velocities of bodies, falling freely by their own weight, are as the times of their falling from reſt.

For ſince the force of gravity is the ſame in all places near the earth's ſurface (by Poſt. 3.) and this is the force by which bodies deſcend. Therefore the falling body is urged by a force which acts conſtantly and equally; and therefore (by Cor. 3. Prop. V.) the velocity generated in the falling body in any time, is as the time of falling.

Cor. 1. *If a body be thrown directly upwards with the ſame velocity it falls with; it will loſe all its motion in the ſame time.*

For the ſame active force will deſtroy as much motion in any time, as it can generate in the ſame time.

Cor. 2. *Bodies deſcending or aſcending gain or loſe equal velocities in equal times.*

PROP. XIII.

In bodies falling freely by their own weight; the ſpaces deſcribed in falling from reſt, are as the ſquares of the times of falling.

For ſince gravity is ſuppoſed to be the ſame in all places near the earth. Therefore the falling

Fig. body will be acted on by a force which is conſtantly the ſame; and therefore (by Prop. VI.) the ſpaces deſcribed, from the beginning of the motion, or ſince their falling from reſt, will be as the ſquares of the times of falling.

Cor. 1. *The ſpaces deſcribed by falling bodies are as the ſquares of the velocities.*

For (Prop. XII.) the velocities are as the times of falling.

Cor. 2. *The ſpaces deſcribed by falling bodies, are in the compound ratio of the times, and the velocities acquired by falling.*

Cor. 3. *If a body falls through any ſpace, and move afterwards with the velocity gained in falling; it will deſcribe twice that ſpace in the time of its falling.*

Cor. 4. *A body projected upward with the velocity it gained in falling, will aſcend to the ſame height it fell from.*

Scholium.

All theſe things would be true if it was not for the reſiſtance of the air, which will retard their motion a little. In very ſwift motions, the reſiſtance of the air has a very great effect in deſtroying the motions of bodies.

PROP. XIV.

4. *If a heavy body* W *be ſuſtained upon an inclined plane* AC, *by a power* F *acting in direction* WF *parallel to the plane; and if* AB *be parallel to the horizon and* BC *perp. to it. Then if the length* AC *denotes the weight of the body, the height* CB *will denote the power at* F *which ſuſtains it, and the baſe* AB *the preſſure againſt the plane.*

Draw BD perpendicular to AC, then CB will be the direction of gravity, DC parallel to WF will be

be the direction of the force at F, and DB the direction of the preſſure (by Prop. IX.) Therefore (by Prop. VIII.) the weight of the body, the power at F, and the preſſure; will be reſpectively as BC, CD and DB. But the triangles ABC, BDC are ſimilar, and therefore BC, CD and DB are reſpectively as AC, CB and BA. Therefore the weight, power and preſſure, are as the lines AC, CB and AB. Fig. 4.

Cor. 1. *The weight of the body, the power* F *that ſuſtains it on the plane, and the preſſure againſt the plane; are reſpectively as radius, the ſine and coſine of the plane's elevation above the horizon.*

For AC, CB and AB are to one another, as radius, ſine of CAB, and ſine of ACB.

Cor. 2. *The power that urges a body* W *down the inclined plane is* $= \frac{CB}{AC} \times$ *weight of* W. *Hence,*

Cor. 3. *If a priſmatic body whoſe length is* AC *lie upon the inclined plane* AC; *it is urged down the plane with a force equal to the weight of the priſm of the length* CB.

Cor. 4. *If there be two planes of the ſame height, and two bodies be laid on them proportional to the lengths of theſe planes; the tendency down the planes will be equal in both bodies.*

PROP. XV.

If AC *be an inclined plane,* AB *the horizon,* BC *perp. to* AB. *And if* W *be a heavy body upon the plane, which is kept there by the power* P *acting in direction* WP. *Draw* BDE *perp. to* WP. *Then the weight* W, *the power* P, *the preſſure againſt the plane; will be reſpectively as* AB, DB *and* AD. 5.

For AB being a horizontal plane is perpendicular to the action of gravity; and BD is perp. to the

Fig. 5. the direction of the power P; and AD is the plane, which is perp. to the direction of the preſſure againſt that plane. Therefore (Cor. 1. Prop. VIII.) the weight of the body, the power P, and the preſſure; are as AB, BD and AD. And if the direction of the power WP be under the plane, the proportion will be the ſame, as long as BD is perpendicular to WP.

Cor. 1. *The weight of the body* W, *is to the power* P *that ſuſtains it* : : *as coſine of the angle of traction* CWP : *to the ſine of the plane's elevation* CAB.

For the weight : power : : AB : BD : : S.ADB or WDE : S.BAD : : coſ. DWE or CWP : S.BAC, where the angle CWP made by the plane and direction of the power is called the angle of traction.

Cor. 2. *Hence it is the ſame thing as to the power and weight, whether the line of direction is above or below the plane, provided the angle of traction be the ſame. For an equal power will ſuſtain the weight in both caſes.*

Cor. 3. *The weight of the body* : *is to the preſſure againſt the plane* : : *as the coſine of the angle of traction* CWP : *to the coſine of* BNP, *the direction of the power above the horizon.*

For the weight W : preſſure : : AB : AD : : S.ADB : S.ABD : : S.ADE : : SNBE : : coſ. EWD or PWC : coſ. BNP.

Cor. 4. *Hence the preſſure againſt the plane is greater when the direction of the power is below the plane, the weight remaining the ſame.*

Scholium.

Although the power has the ſame proportion to the weight, when the angle of traction is the ſame; whether the direction of the power be above or below

below the plane. Yet, ſince the preſſure upon the plane is greater, when the line of direction is below the plane. Therefore in practice, when a weight is to be drawn up hill, if it is to be done by a power whoſe direction is below the plane, the greater preſſure in this caſe will make the carriage ſink deeper into the earth, &c. and for that reaſon will require a greater power to draw it up, than when the line of direction is above the plane. Fig. 5.

PROP. XVI.

If a weight W *upon an inclined plane* AC, *be in equilibrio with another weight* P *hanging freely; then if they be ſet a moving, their perpendicular velocities in that place, will be reciprocally as their quantities of matter.* 6.

Take WA a very ſmall line upon the plane AC; draw AB parallel to the horizon, and BC perp. to it. Draw AF, and WR, BE perpendicular to it; and WT, DV perp. to AB. Let W deſcend through the ſmall line WA upon the plane, then P will aſcend a height equal to AR perpendicularly; and WT will be the perpendicular deſcent of W. The triangles AWR and ADE are ſimilar; and likewiſe the triangles AWT and ADV. Therefore WT : DV : : AW : : AD : : WR : DE. And alternately, WT : WR : : DV : DE; and WR : AR : : DE : AE; therefore WT : AR : : DV : AE : : (by the ſimilar triangles DBV and AEB) DB : AB : : (Prop. XV.) power P : weight W.

Cor. 1. *If any two bodies be in equilibrio upon two inclined planes; their perpendicular velocities will be reciprocally as the bodies.*

Cor. 2. *If two bodies ſuſtain each other in equilibrio, on any planes; the product of one body* × *by its perp.*

Fig. 6. *perp. velocity, is equal to the product of the other body × by its perp. velocity.*

PROP. XVII.

7. *If a heavy body runs down an inclined plane* CA; *the velocity it acquires in any time, moving from reſt; is to the velocity acquired by a body falling perpendicularly in the ſame time; as the height of the plane* CB, *to its length* CA.

The force by which a body endeavours to deſcend on an inclined plane, is to its weight or the force of gravity; as CB to CA (by Prop. XIV.) And as theſe forces always remain the ſame, therefore (Cor. 2. Prop. V.) the velocities generated will be as theſe forces, and the times of acting, directly; and the bodies reciprocally. And ſince the times of acting, and the bodies are the ſame in both caſes, the velocities generated will be as theſe forces; that is, as the height of the plane CB to its length CA.

Cor. 1. *The velocity acquired by a body running down an inclined plane, is as the time of its moving from reſt.*

Cor. 2. *If a body is thrown up an inclined plane, with the velocity it acquired in deſcending; it will loſe all its motion in the ſame time.*

PROP. XVIII.

7. *If a heavy body deſcends down an inclined plane* CA: *the ſpace it deſcribes from the beginning of the motion, is to the ſpace deſcribed by a body falling perpendicularly in the ſame time; as the height of the plane* CB, *to its length* CA.

For the force urging the body down the plane is to the force of its gravity, as CB to CA (by Prop. XIV.)

XIV.) which forces remain conſtantly the ſame. And ſince (Prop. XVII.) the velocities generated in equal times on the plane, and in the perpendicular, are conſtantly as CB to CA; the ſmall particles of ſpace deſcribed with theſe velocities in all the infinitely ſmall portions of time, will ſtill be in the ſame ratio; and therefore the ſums of all theſe ſmall ſpaces, or the whole ſpaces deſcribed from the beginning, will be in the ſame conſtant ratio of CB to CA. Fig. 7.

Cor. 1. *The ſpace deſcribed by a body falling down an inclined plane, in a given time, is as the ſine of the plane's elevation.*

For if CB be given, and alſo the perp. deſcent; that ſpace will be reciprocally as CA, or directly as S.CAB.

Cor. 2. *The ſpaces deſcribed by a body deſcending from reſt, down an inclined plane, are as the ſquares of the times.*

Cor. 3. *If* BD *be drawn perp. to the plane* CA; *then in the time a body falls perpendicularly through the height* CB, *another body will deſcend through the ſpace* CD *upon the plane.*

For by ſimilar triangles CA : CB : : CB : CD.

PROP. XIX.

If AC *is an inclined plane, the time of a body's deſcending through the plane* CA, *is to the time of falling perpendicularly through the height of the plane* CB, *as the length of the plane* CA *to the height* CB. 7.

For if BD is perp. to CA, then (Cor. 2. Prop. XVIII.) ſpace CD : ſpace CA : : ſquare of the time in CD : ſquare of the time in CA : : that is, as the ſquare of the time of deſcending perpendicularly

Fig. 7. cularly in CB (Cor. 3. laſt) : ſquare of the time in CA. But CD : CA : : CB^2 : CA^2. Therefore CB^2 : CA^2 : ſquare of the time in CB : ſquare of the time in CA. And CB : CA : : time in CB : time in CA.

Cor. 1. *If a body be thrown upwards on the plane with the velocity acquired in deſcending; it will in an equal time aſcend to the ſame height.*

Cor. 2. *The times wherein different planes, of the ſame height, are paſſed over; are as the lengths of the planes.*

Let the planes be CA, CF. Then time in CA : time in CB : : CA : CB. And the time in CB : time in CF : : CB : CF. Therefore, *ex equo*, time in AC : time in CF : : CA : CF.

PROP. XX.

8. *If a body falls down an inclined plane, it acquires the ſame velocity as a body falling perpendicularly through the height of the plane.*

Let the body run down the plane CA whoſe height is CB. Draw DF parallel to AB, and infinitely near it. Then the velocities in DA and FB, may be looked upon as uniform. Now (Prop. XIX.) the times of deſcribing CA and CB, will be as CA and CB. Likewiſe the times of deſcribing CD and CF, will be as CD and CF; that is, as CA and CB. And by diviſion, the difference of the times, or the times of deſcribing DA and FB, will alſo be as CA and CB; that is, as DA and FB. But (from Prop. III.) the velocities are equal when the ſpaces are as the times of deſcription. Therefore velocity at A is equal to the velocity at B.

Cor.

Cor. 1. *The velocities acquired, by bodies deſcending on any planes, from the ſame height to the ſame horizontal line, are equal.* Fig. 8.

Cor. 2. *If the velocities be equal at any two equal altitudes* D, F; *they will be equal at all other equal altitudes* A, B.

Cor. 3. *Hence alſo, if ſeveral bodies be moving in different directions, through any ſpace contained between two parallel planes; and be acted on by any force, which is equal at equal diſtances from either plane. Then if their velocities be equal at entering that ſpace; they will alſo be equal at emerging out of it.*

For dividing that ſpace into infinitely ſmall parts by parallel planes. Then the force between any two planes may be ſuppoſed uniform; and ſuppoſing DF, AB to repreſent two of theſe planes, then (by Cor. 2.) the velocities at D and F being equal; the velocities at A and B will be equal; that is, the velocities at entering the firſt part of ſpace being equal, the velocities at emerging out of it, or at entering the ſecond ſpace will be equal. And for the ſame reaſon the velocities at entering the ſecond ſpace being equal, thoſe at emerging out of it into the third, will be equal. And conſequently the velocities at entering into, and emerging out of the third, fourth, fifth, &c. to the laſt will be equal reſpectively.

PROP. XXI.

If a body falls from the ſame hight, through any number of contiguous planes AB, BC, CD; *it will at laſt gain the ſame velocity as a body falling perpendicularly from the ſame height.* 9.

Let FH be a horizontal line, FD perp. to it. Produce the planes BC, DC to G and H. Then

(Cor.

Fig. 9. (Cor. 1. Prop. XX.) the velocity at B is the ſame whether the body deſcend through AB or GB. And therefore the velocity at C will be the ſame, whether the body deſcends through ABC or through GC, and this is the ſame as if it had deſcended through HC. And conſequently it will have the ſame velocity at D, in deſcending through the planes ABCD, as in deſcending through the plane HD; that is, (Prop. XX.) as it has in deſcending through the perpendicular FD.

Cor. 1. *Hence a body deſcending along any curve ſurface, will acquire the ſame velocity, as if it fell perpendicularly through the ſame height.*

For let the number of planes be increaſed, and their length diminiſhed *ad infinitum*, and then ABCD will become a curve. And the velocity acquired by deſcending through theſe infinite planes; that is, through the curve ABCD, will be the ſame as in falling perpendicularly through FD.

Cor. 2. *If a body deſcends in a curve, and another deſcends perpendicularly from the ſame height. Their velocities will be equal at all equal altitudes.*

Cor. 3. *If a body, after its deſcent in a curve, ſhould be directed upwards with the velocity it had gained; it will aſcend to the ſame height from which it fell.*

For ſince gravity acts with the ſame force whether the body aſcends or deſcends, it will deſtroy the velocity in the aſcent, in the ſame time it did generate it in the deſcent.

Cor. 4. *The velocity of a body deſcending in any curve, is as the ſquare root of the height fallen from.*

For it is the ſame as in falling perpendicularly; and in falling perpendicularly, it is as the ſquare root of the height.

Cor.

Cor. 5. *If a body, in moving through any ſpace* ED, *be acted on uniformly by any force; its velocity at emerging out of it at* D, *will be equal to the ſquare root of the ſum of the ſquares of the velocity at* E *in entering of it, and of the velocity acquired in falling from reſt through that ſpace* ED. *And this holds, whether the body moves perpendicularly or obliquely.* Fig. 9.

For let the body enter the ſpace ED at E, with the velocity acquired in falling through FE. Then (Prop. XIII. Cor. 1.) the ſquare of the velocity at E will be as FE; and the ſquare of the velocity at D, as FD; and the ſquare of the velocity at D falling from E, will be as ED. But FD = FE + ED; therefore the ſquare of the velocity at D (falling through FD) = ſquare of the velocity at E + ſquare of the velocity at D (falling through ED.) And (Cor. 1. of this Prop.) the velocity will be the ſame whether the body deſcends perpendicularly or obliquely.

PROP. XXII.

The times of bodies deſcending through two ſimilar parts of ſimilar curves, placed alike, are as the ſquare roots of their lengths. 10.

Let ABCD and *abcd* be two ſimilar curves, and ſuppoſe BC and *bc* to be infinitely ſmall, and ſimilar to the whole; that is, ſo that BC : *bc* : : AB : *ab*. Draw FA parallel to the horizon, and HB, *hb* perp. to it. Then if two bodies deſcend from A and *a* (Cor. 4. Prop. XXI.) the velocities at B and *b* will be as $\sqrt{HB}$ and $\sqrt{hb}$; that is, as $\sqrt{AB}$ and $\sqrt{ab}$, becauſe AB, *ab* are ſimilar parts. Therefore (Prop. III. Cor. 1.) the times of deſcribing BC and *bc*, are as $\frac{BC}{\sqrt{AB}}$ and $\frac{bc}{\sqrt{ab}}$; that is,

Fig. 10. as $\frac{AB}{\sqrt{AB}}$ and $\frac{ab}{\sqrt{ab}}$, or as $\sqrt{AB}$ and $\sqrt{ab}$; that is, as $\sqrt{AD}$ and $\sqrt{ad}$, becauſe the curves are ſimilarly divided in B and *b*. After the ſame manner the times of deſcribing any other two ſimilar parts as BC, *bc*, will be as $\sqrt{AD}$ and $\sqrt{ad}$. Therefore by compoſition the times of deſcribing all the BC's, and all the *bc*'s will be as $\sqrt{AD}$ and $\sqrt{ad}$. That is, the time of deſcribing the curve AD to the time of deſcribing the curve *ad*, is as $\sqrt{AD}$ to $\sqrt{ad}$.

Cor. *If two bodies deſcend through two ſimilar curves* ABD, *and abd; the axes of the curves* FD, F*d* *are as the ſquares of the times of their deſcending.*

For $\sqrt{FD} : \sqrt{Fd} :: \sqrt{AD} : \sqrt{ad}$: : time of deſcending through ABD : time of deſcending through *abd*. And FD, F*d*, are as the ſquares of the times.

PROP. XXIII.

11. *A body will deſcend through any chord of a circle, in the ſame time that another deſcends perpendicularly through the diameter.*

Draw the diameter AB perpendicular to the horizon, and the chords CA, CB. Then ſince BC is perpendicular to AC, therefore (Prop. XVIII. Cor. 3.) the time of deſcending through the chord AC is equal to the time of falling through AB.

Draw CD parallel to AB, and DB parallel to CA, then is CD equal to AB. And by reaſon of the parallels, the angle DBC = angle BCA = a right angle. Then ſince DB is perp. to CB, therefore (Cor. 3. Prop. XVIII.) a body will deſcend through the inclined plane CB, in the ſame time that another falls through CD, or, which is the ſame thing, through its equal AB.

Cor.

Cor. 1. *Hence the times of deſcending through all the chords of a circle drawn from* A *or* B, *are equal among themſelves.* Fig. 11.

Cor. 2. *The velocity gained by falling through the chord* CB, *is as its length* CB.

For the velocity gained in falling through CB is the ſame as is gained by falling through EB; and that velocity is to the velocity gained by falling through AB, as $\sqrt{BE}$ to $\sqrt{AB}$ (by Cor. 1. Prop. XIII.); that is, as BC to BA. Therefore if the given velocity in falling down AB be repreſented by AB, the velocity gained in falling down CB will be repreſented by CB; and ſo that in any other chord, by its length.

PROP. XXIV.

If a pendulum vibrates in the ſmall arch of a circle; the time of one vibration, is to the time of a body's falling perpendicularly through half the length of the pendulum; as the circumference of a circle, to the diameter. 12.

If a pendulum ſuſpended by a thread, &c. be made to vibrate in any curve; it is the ſame thing as if it deſcended down a ſmooth poliſhed body made in the form of that curve. For the motions, velocities, and times of moving will be the ſame in both.

Let OD or OE be the pendulum vibrating in the arch ADC, whoſe radius is OD. Let OD be perp. to the horizon, and take the arch E*e* infinitely ſmall, and draw ABC, EFG, *efg*, perp. to OD; and draw the chord AD. About BD deſcribe the ſemicircle BGD. Draw *er* and G*s* perp. to EG.

Put t = time of deſcending through the diameter 2OD, or through the chord AD. Then the velocities

Fig. 12. gained by falling through 2OD, and by the pendulum's deſcending through the arch AE, will be as $\sqrt{2OD}$ and $\sqrt{BF}$. And the ſpace deſcribed in the time t, after the fall through 2OD, is 4OD. But the times are as the ſpaces, divided by the velocities. Therefore, $\frac{4OD}{\sqrt{2OD}}$ or $2\sqrt{2OD} : t$ (time of its deſcription) $:: \frac{Ee}{\sqrt{BF}}$: time of deſcribing $Ee = \frac{t \times Ee}{2\sqrt{2OD \times BF}}$.

But by the ſimilar triangles OEF, E*er*; and KGF, G*gs*; we ſhall have $\frac{EF}{OD} \times Ee = er = Ff = Gs = \frac{FG}{KD} \times Gg$. Whence $Ee = \frac{OD \times FG}{KD \times EF} \times Gg$. Therefore the time of deſcribing $Ee = \frac{t \times OD \times FG \times Gg}{2KD \times EF\sqrt{BF \times 2OD}} = \frac{t \times OD \times \sqrt{BF \times FD} \times Gg}{2KD\sqrt{BF} \times \sqrt{DO+OF \times FD} \times \sqrt{2OD}} = \frac{t \times \sqrt{OD} \times Gg}{2KD \times \sqrt{DO+OF} \times \sqrt{2}} = \frac{t \times \sqrt{2OD}}{4KD \times \sqrt{DO+OF}} \times Gg = \frac{t \times \sqrt{2OD}}{2BD \times \sqrt{2OD-DF}} \times Gg$. But DF, in its mean quantity for all the arches Gg, is nearly equal to DK. Therefore the time of deſcribing $Ee = \frac{t \times \sqrt{2OD}}{2BD\sqrt{2OD-DK}} \times Gg$. Whence the time of deſcribing the arch $AED = \frac{t \times \sqrt{2OD}}{2BD\sqrt{2OD-DK}} \times BGD$. And the time of deſcribing the whole arch ADC, or the time of one oſcillation is $= t \times$

$\frac{t \times \sqrt{2OD}}{2BD\sqrt{2OD - DK}} \times 2BGD$. But when the arch ADC is exceeding ſmall, DK vaniſhes, and then the time of oſcillation in a very ſmall arch is $= \frac{t \times \sqrt{2OD}}{2BD\sqrt{2OD}} \times 2BGD = \frac{1}{2}t \times \frac{2BGD}{BD}$. But if t be the time of deſcending through 2OD, $\frac{1}{2}t$ is the time of deſcending through $\frac{1}{2}$OD. And therefore BD the diameter, is to 2BGD the circumference; as the time of falling through half the length of the pendulum, to the time of one vibration. Fig. 12.

Cor. 1. *In a ſmall arch* AED, *the time of deſcending through the chord* AD, *is to the time of deſcending through the arch* AED; *as the diameter* BD, *to* $\frac{1}{4}$ *the circumference.*

For the time of deſcending through the arch $AED = t \times \frac{BGD}{2BD}$; therefore BD : $\frac{1}{2}$BGD : : t : time in AED.

Cor. 2. *All the vibrations of the ſame pendulum, in arches not very large, are performed nearly in the ſame time.*

Cor 3. *If* KD *be biſected in* L, *and* T *be* = *time of vibration in a very ſmall arch. Then* $T + \frac{KL}{DO + OK} \times T$ *will be the time of vibration in any arch* ADC, *nearly.*

For we found the time of vibration in $ADC = \frac{t \times BGD}{BD} \times \sqrt{\frac{2OD}{2OD - DK}} = T \times \sqrt{\frac{2OD}{OD + OK}}$; and the three lines DO + OK, DO + OL, and DO + OD are in arithmetical progreſſion; but ſince KD is very ſmall, they are nearly in geometri-

Fig. 12. cal progreſſion; whence $\sqrt{\frac{2OD}{DO+OK}} = \frac{DO+OL}{DO+OK}$.

Therefore the time of vibration $= T \times \frac{DO+OL}{DO+OK}$

$= T \times \frac{DO+OK+KL}{DO+OK} = T + T \times \frac{KL}{DO+OK}$.

Cor. 4. *Hence a falling body will deſcend through a ſpace of* 16 *feet and* 1 *inch, in a ſecond of time.*

For by obſervation, a pendulum 39.13 inches long will ſwing ſeconds. And $t \times \frac{BGD}{BD} = 1$ ſecond, and $\frac{BD}{BGD} = t$, or $\frac{2}{3.1416}$ = time of falling through 2×39.13. Whence (Prop. XIII.) $\frac{4}{3.1416^2} : 2 \times 39.13 :: 1^2 : \frac{39.13}{2} \times 3.1416^2 = 193.096$ inches $= 16.09$ feet.

Cor. 5. *The ſpace deſcribed by a falling body in the time of one vibration : is to half the length of the pendulum : : as the ſquare of the circumference : to the ſquare of the diameter.*

For the ſpaces being as the ſquares of the times, the ſpace fallen : $\frac{1}{2}$ length of the pendulum : : as the ſquare of the time of vibration : to the ſquare of the time of falling through $\frac{1}{2}$ length of the pendulum : : (by this Prop.) as the ſquare of the circumference : to the ſquare of the diameter.

PROP. XXV.

The lengths of two pendulums, deſcribing ſimilar arches, are as the ſquares of the times of vibration.

For (Prop. XXII.) the times of deſcending through two ſimilar curves, are as the ſquare roots of the lengths of the curves; that is, as the ſquare roots

roots of the lengths of the pendulum, their centers Fig.
being alike ſituated. Therefore the lengths of the pendulums are as the ſquares of the times of vibrating.

Cor. 1. *The times of vibration of pendulums in ſmall arches of circles, are as the ſquare roots of the lengths of the pendulums.*

For if the arches are ſimilar, the times of vibration are in that proportion. And (Prop. XXIV. Cor. 2.) if the arches are ſmall, though not ſimilar, the vibration will be the ſame as before.

Cor. 2. *The velocity of a pendulum at its loweſt point, is as the chord of the arch it deſcends through.*

For the velocity at the loweſt point is equal to the velocity gained in deſcending through the chord; for they are both of them the ſame as a body acquires by falling through their common altitude. And (Prop. XXIII, Cor. 2.) the velocity gained in falling through the chord, is as the length of the chord. Therefore the velocities of a pendulum in different arches, are in the ſame ratio.

PROP. XXVI.

Pendulums of the ſame length vibrate in the ſame time, whether they be heavier or lighter.

For let the two pendulums P, p, be of the ſame length; they will each of them fall through half the length of the pendulum in the ſame time. For (Cor. 2. Prop. V.) the velocity generated in any given time, is as the force directly and matter reciprocally. But in the two pendulums, the forces that generate their motions, are their weights, which are as their quantities of matter. Whence we have the velocity of P, to the velocity of p; as $\frac{P}{P}$ to $\frac{p}{p}$ or as 1 to 1; and therefore equal velocities are generated in the ſame time. Conſequently,

Fig. equal ſpaces will be deſcribed in the ſame time, and therefore they will fall through half the length of one of them in an equal time. And therefore (Prop. XXIV.) their vibrations will be performed in the ſame time.

Cor. *Hence all bodies, whether greater or leſſer, heavier or lighter, near the earth's ſurface, will fall through equal ſpaces in equal times; abating the reſiſtance of the air.*

Becauſe they are as much retarded by their matter, as accelerated by their weight. The weight and the matter being exactly proportional to one another.

PROP. XXVII.

The lengths of pendulums vibrating in the ſame time, in different places of the world, will be as the forces of gravity.

For (by Prop. V. Cor. 2.) the volocity generated in any time is as the force of gravity directly, and the quantity of matter reciprocally. And the matter being ſuppoſed the ſame in both pendulums, the velocity is as the force of gravity; and the ſpace paſſed through in a given time, will be as the velocity; that is, as the gravity. Therefore, if any two ſpaces be deſcended through in any time, and two pendulums be made, whoſe lengths are double theſe ſpaces; theſe pendulums (by Prop. XXIV.) will vibrate in equal times; therefore the lengths of rhe pendulums, being as the ſpaces fallen through in equal times, will be as the forces of gravity.

Cor. 1. *The times wherein pendulums of the ſame length will vibrate, by different forces of gravity, are reciprocally as the ſquare roots of the forces.*

For (Cor. 2. Prop. V.) when the matter is given, the velocity generated is as the force × by the time.

And

And (Prop. VI.) the ſpace deſcended through by Fig.
any force, is as the force and ſquare of the time. Let theſe ſpaces be the lengths of the pendulums. Then the lengths of the pendulums are as the forces and the ſquares of the times of falling through them. But (Prop. XXIV.) the times of falling through them are in a given ratio to the times of vibration; whence the lengths of pendulums are as the forces and the ſquares of the times of vibration; therefore when the lengths are given, the forces will be reciprocally as the ſquares of the times; and the times of vibration reciprocally as the ſquare roots of the forces.

Cor. 2. *The lengths of pendulums in different places, are as the forces of gravity, and the ſquares of the times of vibration.*

This is proved under Cor. 1. Hence,

Cor. 3. *The times wherein pendulums, of any length, perform their oſcillations; are as the ſquare roots of their lengths directly, and the ſquare roots of the gravitating forces reciprocally.*

Cor. 4. *The forces of gravity in different places, are as the lengths of pendulums directly, and the ſquares of the times of vibration reciprocally.*

PROP. XXVIII. *Prob.*

To find the length of a pendulum, that ſhall make any number of vibrations in a given time.

Reduce the given time into ſeconds, then ſay, as the ſquare of the number of vibrations given : to the ſquare of this number of ſeconds : : ſo is 39.13 : to the length of the pendulum ſought, in inches.

Ex. Suppoſe it makes 50 vibrations in a minute; here a minute is = 60 ſeconds; then,

As 2500 (the ſquare of 50) : 3600 (the ſquare of

Fig. of 60) : : 39.13 : to the length $= \frac{3600 \times 39.13}{2500}$

$= \frac{140868}{2500} = 56.34$ inches, the length required.

If it be required to find a pendulum that ſhall vibrate ſuch a number of times in a minute; you need only divide 140868, by the ſquare of the number of vibrations given, and the quotient will be the length of the pendulum.

This practice is deduced from Prop. XXV. for let p be the length of the pendulum, n the number of vibrations, t the time they are to be performed in. Then $39.13 : 1^2 : : p : \frac{p}{39.13}$ = ſquare of the time of one vibration, and $\sqrt{\frac{p}{39.13}}$ = time of one vibration; then if t be divided by $\sqrt{\frac{p}{39.13}}$ it will give n; that is, $t\sqrt{\frac{39.13}{p}} = n$, whence $tt \times 39.13 = nnp$, and $nn : tt : : 39.13 : p$.

If the pendulum is a thread with a little ball at it, then the diſtance between the point of ſuſpenſion and the center of the ball is eſteemed the length of the pendulum. But if the ball be large, ſay as the diſtance between the point of ſuſpenſion, and the center of the ball, is to the radius of the ball; ſo the radius of the ball to a third proportional. Set $\frac{2}{5}$ of this from the center of the ball downward, gives the center of oſcillation. Then the whole diſtance from the point of ſuſpenſion to this center of oſcillation, is the true length of the pendulum.

13. If the bob of the pendulum be not a whole ſphere, but a thin ſegment of a ſphere, as AB, as in muſt clocks; then to find the center of oſcillation, ſay as the diſtance between the point of ſuſpenſion, and the middle of the bob, is to half the breadth

breadth of the bob; ſo half the breadth of the bob, to a third proportional. Set one third of this length from the middle of the bob downwards, gives the center of oſcillation. Then the diſtance between the centers of ſuſpenſion and oſcillation, is the exact length of the pendulum. Fig.

PROP. XXIX. *Prob.*

Having the length of a pendulum given; to find how many vibrations it ſhall make in any given time.

Reduce the time given into ſeconds, and the pendulum's length into inches; then ſay, as the given length of the pendulum : to 39.13 : : ſo is the ſquare of the time given : to the ſquare of the number of vibrations, whoſe ſquare root is the number ſought.

Example. Suppoſe the length of the pendulum is 56.34 inches, to find how often it will vibrate in a minute.

1 minute = 60 ſeconds. Then 56.34 (the length of the pendulum) : 39.13 : : 3600 (the ſquare of 60) : to the ſquare of the number of vibrations $= \frac{3600 \times 39.13}{56.34} = \frac{140868}{56.34} = 2500$, and $\sqrt{2500} = 50 =$ the number of vibrations ſought.

If the time given be a minute, you need only divide 140868 by the length, and extract the root of the quotient for the number of vibrations.

This is the reverſe of the laſt problem, therefore, ſuppoſing as before in that problem, we have $tt \times 39.13 = nnp$; therefore $p : 39.13 : : tt : nn$.

They that would ſee a further account of the motions of bodies upon inclined planes, the vibrations of pendulums, and the motion of projectiles; may conſult my large book of Mechanics, where they will meet with full ſatisfaction.

SECT.

Fig.

SECT. III.

Of the Center of Gravity; the equilibrium of beams of timber; the directions and quantities of the forces neceſſary to ſuſtain them.

PROP. XXX.

A BODY cannot deſcend or fall downwards, except only when it is in ſuch a poſition, that by its motion, its center of gravity deſcends.

14. Let the body A ſtand upon the horizontal plane BK, and let C be its center of gravity; draw CD perpendicular to the plane BK. And let the body be ſuſpended at the point C, upon the perpendicular line CD. Then (def. 12) it will remain unmoved upon the line CD. And as CD is perp. to the horizon, it has no inclination to move one way more than another, therefore it will move no way but remain at reſt. Take away the line CD, and let the body be ſupported by the line BC; if BC be fixed, the body will remain at reſt on the line BC. But if BC be movable about B, the body ſuſpended at C, will endeavour to move with its center of gravity downwards along the arch CE, about B as a center, towards N. And for the ſame reaſon the body will endeavour to fall the contrary way, moving about the point N; I ſay, this will be the caſe when CD is ſituated between B and N. But theſe two motions being contrary to one another, will hinder each other's effects; and the body will be ſuſtained without falling.

Again,

Again, let the body F be ſuſpended with its center of gravity I upon the perpendicular IH. As this line has no inclination to move to any ſide, it will therefore remain at reſt. Take away the line IH, and let the center of gravity I be ſuſpended on the line IG, then the body will endeavour to deſcend along the arch IK, for the higheſt point of the arch is in the perpendicular erected at G. For the ſame reaſon if the body be ſuſpended on the line OI, it will endeavour to deſcend towards K, about the center O; now as both theſe motions tend the ſame way, and there is nothing to oppoſe them; the body muſt fall towards K. In both theſe caſes it is plain, that when the center of gravity by its motion, deſcends, the body will fall; but if not, the body will be ſupported without falling. Fig. 14.

Cor. 1. *If a body ſtands upon a plane, and a line be drawn from the center of gravity perpendicular to the horizon; if this line fall within the baſe on which the body ſtands, it will be ſupported without falling. But if the perpendicular falls without the baſe, the body will fall.*

For when the perpendicular falls within the baſe, the body can be moved no manner of way, but the center of gravity will riſe. And when the perpendicular falls without the baſe, towards any ſide; if the body be moved towards that ſide, the center of gravity deſcends; and therefore the body will fall that way.

Cor. 2. *If a perpendicular drawn from the center of gravity perp. to the horizon, fall upon the extremity of the baſe, the body may ſtand; but the leaſt force whatever, will cauſe it to fall that way. And the nearer the perp. is to any ſide, the eaſier it will be made to fall, or is ſooneſt thruſt over. And the nearer the perp. is to the middle of the baſe, the firmer the body ſtands.*

Cor.

Fig. 14. Cor. 3. *Hence if the center of gravity of a body be ſupported, the whole body is ſupported. And the place of the center of gravity muſt be deemed the place of the body; and is always in a line drawn perpendicular to the horizon, through the center of gravity.*

Cor. 4. *Hence all the natural actions of animals may be accounted for from the properties of the center of gravity.*

When a man endeavours to walk, he ſtretches out his hind leg, and bends the knee of his fore leg, by which means his body is thruſt forward, and the center of gravity of his body is moved forward beyond his feet; then to prevent his falling, he immediately takes up his hind foot, and places it forward beyond the center of gravity; then he thruſts himſelf forward, by his leg which now is the hindmoſt, till his center of gravity be beyond his fore foot, and then he ſets his hind foot forward again; and thus he continues walking as long as he pleaſes.

In ſtanding, a man having his feet cloſe together cannot ſtand ſo firmly, as when they are at ſome diſtance; for the greater the baſe, the firmer the body will ſtand; therefore a globe is eaſily moved upon a plane, and a needle cannot ſtand upon its point, any otherwiſe than by ſticking it into the plane.

When a man is ſeated in a chair, he cannot riſe till he thruſts his body forward, and draws his feet backward, till the center of gravity of his body be before his feet; or at leaſt upon them; and to prevent falling forward, he ſets one of his feet forward, and then he can ſtand, or ſtep forward as he pleaſes.

All other animals walk by the ſame rules; firſt ſetting one foot forward, that way the center of gravity leans, and then another.

In

In walking up a hill, a man bends his body forward, that the center of gravity may lie forward of his feet; and by that means he prevents his falling backwards. Fig.

In carrying a burthen, a man always leans the contrary way that the burthen lies; ſo that the center of gravity of the whole of his body and the burthen, may fall upon his feet.

A fowl going over an obſtacle, thruſts his head forward, by that means moving the center of gravity of his whole body forwards; ſo that by ſetting one foot upon the obſtacle, he can the more eaſily get over it.

PROP. XXXI.

In any two bodies A, B, *the common center of gravity* C, *divides the line joining their centers, into two parts, which are reciprocally as the bodies*; AC : BC : : B : A. 15.

Let the line ACB be ſuppoſed an inflexible lever; and let the lever and bodies be ſuſpended on the point C. Then let the bodies be made to vibrate about the immovable point C; then will A and B deſcribe two arches of circles about the center C, and theſe arches will be as the velocities of the bodies, and theſe arches are alſo as the radii of the circles AC and BC. Therefore their velocities are as the radii. Whence, velocity of A : velocity of B : : AC : CB : : (by ſuppoſition) B : A. Therefore A × velocity of A = B × velocity of B. Whence (Prop. II.) the quantities of motion of the bodies A and B are equal, and (Ax. 9.) therefore they cannot move one another, but muſt remain at reſt; and conſequently (def. 12.) C is the center of gravity of A and B.

Cor.

Fig. 15. Cor. 1. *The products of each body multiplied by its diſtance from the common center of gravity, are equal.* CA × A = CB × B.

Cor. 2. *If a weight be laid upon* C, *a point of the inflexible lever* AB, *which is ſupported at* A *and* B; *the preſſure at* A *to the preſſure at* B, *will be as* CB *to* CA.

For let the bodies A, B, be both placed in C; then (Cor. 3. Prop. XXX.) ſince it is the ſame thing whether the bodies be at A and B, or both of them at C, their center of gravity; therefore the preſſures at A and B will be the ſame in both caſes. But when they are at A and B, upon the lever ACB, their preſſures are A and B, being the ſame with the weights; therefore when they are both at C, the preſſures at A and B will ſtill be A and B. Therefore (Cor. 1.) ſince it is CA × A = CB × B; therefore CA : CB : : B : A : : preſſure at B : preſſure at A.

PROP. XXXII.

15. *If there be three or more bodies, and if a line be drawn from one body* E *to the center of gravity of the reſt* C. *Then the center of gravity of all the bodies divides the line* CE, *in two parts in* D, *which are reciprocally as the body* E *to the ſum of all the other bodies.* CD : DE : : E : A + B *&c.*

For ſuppoſe AB and CE to be two inflexible lines; and let the body W = A + B &c. and let W be placed in the center of gravity C. Then by the laſt Prop. CD : DE : : E : W or A + B &c.

Cor. *The body* E × DE *the diſtance from the common center of gravity, is equal to the ſum of the bodies* A + B *&c.* × *by* DC *the diſtance of their center from the common center of gravity.*

PROP.

Fig.

PROP. XXXIII.

If A, B, *be two bodies,* C *their center of gravity.* 16.
F *any point in the line* AB. *Then will* $FA \times A + FB \times B = FC \times \overline{A+B}$.

For (Cor. 1. Prop. XXXI.) $CA \times A = CB \times B$; that is, $\overline{FA-FC} \times A = \overline{FC-FB} \times B$; whence, by tranſpoſition $FA \times A + FB \times B = FC \times \overline{A+B}$.

Cor. *Hence the bodies* A *and* B *have the ſame force to turn the lever* AF *about the point* F, *as if they were both placed in* C, *their center of gravity.*

PROP. XXXIV.

If ſeveral bodies, A, B, E *&c. be placed on an* 17.
inflexible ſtraight lever; and if D *be their common center of gravity; and if* F *be any point in the line* AE, *then* $FA \times A + FB \times B + FE \times E$ *&c.* $= FD \times \overline{A+B+E}$ *&c.*

For if $A + B = W$, then $FA \times A + FB \times B + FE \times E = FC \times \overline{A+B} + FE \times E = FC \times W + FE \times E =$ (Prop. XXXIII.) $FD \times \overline{W+E} = FD \times \overline{A+B+E}$, in the three bodies A, B, E. And after the ſame manner, if there be four bodies, put $W = A + B + E$, and it will be proved the ſame way, that the ſum of all the products, $FA \times A + FB \times B$ &c. = diſtance of the common center of gravity × by all the four. And ſo on for more bodies.

Cor. *The ſame Prop. will hold good, when the bodies are not in the line* AF, *but any where in the perpendiculars paſſing through the points* A, B, E *&c.*

Fig.

PROP. XXXV.

17. *If there be any number of bodies* A, B, E, *&c. either placed in the line* AF, *or any way in the perpendiculars paſſing through* A, B, E. *And if* D *be the center of gravity of all the bodies; and* F *be any point in the line* AF. *Then the diſtance of the center of gravity* $FD = \frac{FA \times A + FB \times B + FE \times E}{A + B + E}$.

For whether the bodies be in the points A, B, E, or in the perpendiculars, it will be (by Prop. XXXIV. and Cor.) that $FA \times A + FB \times B + FE \times E = FD \times \overline{A + B + E}$. Whence $FD = \frac{A \times FA + B \times FB + E \times FE}{A + B + E}$ = ſum of all the products of each body multiplied by its diſtance, divided by the ſum of the bodies.

Cor. 1. *If a ſingle body only was placed on the lever* AF; *then the diſtance of the center of gravity of that body is equal to the ſum of the products of all the particles of the body, each multiplied by its diſtance from a given point* F, *and divided by the body.*

For if A, B, E &c. are ſeveral particles of the body, then A + B + E &c. = the body; and $FD = \frac{A \times FA + B \times FB + E \times FE}{\text{body}}$.

Cor. 2. *If there be ſeveral bodies* A, B, E, *&c. placed upon the lever* AF. *They act with the ſame force in turning the lever about any given point* F, *as if they were all placed in* D *the common center of gravity of all the bodies.*

SCHOLIUM.

If any of the bodies be placed on the contrary ſide of F, *their reſpective products will be negative.*

For

For they act the contrary way in turning the lever about. Fig. 17.

PROP. XXXVI.

If ſeveral bodies A, B, E, G, H, *be placed on the lever* AH, *and* F *be the center of gravity of all their weights. Then* $FA \times A + FB \times B + FE \times E = FG \times G + FH \times H$. 18.

For let the lever be ſuſpended on the point F, then the two ends will be in equilibrio, as F is the center of gravity. Let D be the center of gravity of A, B, E; and I the center of gravity of G, H. Then (Cor. Prop. XXXIII.) it is the ſame thing whether the bodies on one ſide be placed at A, B, E, or all of them in the point D. And whether thoſe at the other end be placed at G, H; or all of them at I. But ſince F is the center of gravity, $DF \times \overline{A+B+E} = FI \times \overline{G+H}$, and therefore $A \times AF + B \times BF + E \times EF = G \times GF + H \times HF$ (by Prop. XXXIV.)

Cor. 1. *If ſeveral bodies* A, B, E, G, H, *be placed on an inflexible lever, and if* $A \times FA + B \times FB + E \times FE = G \times FG + H \times FH$. *Then* F *is the center of gravity of all the bodies.*

For no other point will anſwer the equation.

Cor. 2. *If ſeveral bodies* A, B, E, G, H, *be placed upon a lever* AH, *or ſuſpended at theſe points by ropes; and if* $A \times FA + B \times FB + E \times FE = G \times FG + H \times FH$; *they will be in equilibrio upon the point* F.

This appears by Def. 12, and F is center of gravity.

Fig.

PROP. XXXVII.

19. *If a heavy body* AB, *ſuſpended by two ropes* AC, BD, *remains at reſt; a right line perpendicular to the horizon, paſſing through the interſection* F *of the ropes; will alſo paſs through the center of gravity* G, *of the body.*

If AC and BD be produced to F where they interſect; then (Ax. 12) it is the ſame thing whether the force acting in direction AC be applied to C or F; and whether the force acting in direction BD be applied to the point D or F. Suppoſe then that they both act at F, and then it is the ſame thing, as if the body was ſuſpended at F by the two ſtrings AF, BF. And ſince the body is at reſt, therefore (Ax. 7.) the body, that is, the center of gravity G, is at the loweſt place it can get; and therefore is in the plumb line FG. For if the body be made to vibrate, the center of gravity G will deſcribe an arch of a circle, of which G (being in the perp. FG) is the loweſt point.

Cor. 1. *Hence if* GN *be drawn parallel to* AC; *then the weight of the body, the forces acting at* C *and* D, *are reſpectively as* FG, GN, *and* FN.

This is evident by Prop. VIII.

Cor. 2. *If a heavy body* AB, *be ſupported by two planes,* IKL, *and* EHG, *at* H *and* K; *and* HF *and* KF *be drawn perpendicular to theſe planes; and if* FG *be drawn from the interſection* F, *perp. to the horizon, it will paſs through the center of gravity* G, *of the body.*

For ſince the body is ſuſtained by theſe planes, therefore the planes re-act againſt the body (by Prop. IX.) in the directions HF, KF perpendicular to theſe planes. Therefore it is the ſame thing

as

as if the body was ſuſtained by the two ropes HF, KF. For the direction and quantities of the forces, acting at H and K are the ſame in both caſes. And further, if the body be made to vibrate round F, the points H, K will deſcribe two arches of circles, coinciding with the touching planes at H, K; therefore moving the body up and down the planes, will be juſt the ſame thing, as making it vibrate in the ropes, HF and KF; and conſequently, the body can reſt in neither caſe, but when the center of gravity G is in the perpendicular FG. Fig. 19.

Scholium.

If any body ſhould deny the truth of this Prop. or its corollaries, againſt the cleareſt force of demonſtration. It lies upon them to ſhew where the demonſtration fails, or what ſtep or ſteps thereof do not hold good, or are not truly deduced from the foregoing. If they cannot do this, what other reaſons they may aſſign, can ſignify nothing at all to the purpoſe. And if ſuch perſon, ignorant of the laws of nature, and the reſolution of forces, would object againſt this practice, of ſubſtituting planes perpendicular to the lines or cords ſuſtaining any weights, inſtead of theſe cords, let him firſt read Sir *I. Newton*'s *Principia*, Cor. 2. to the laws of nature, where he will ſee this practice exemplified, and then make his objections.

And for the ſake of ſuch perſons as underſtand not how to apply the method of compoſition and reſolution of forces, I ſhall add a few problems to prevent their being miſled by the raſh judgment of ſome people, who having brought out falſe ſolutions to ſome problems by their own ill management, condemn the method as erroneous; when the fault really lies in their own ignorance, and not at all in the method itſelf.

Fig. 20.

PROP. XXXVIII. *Prob.*

To determine the position of a beam CD, *movable about the end* C, *and sustained by a given weight* Q, *hanging at a rope* QAD, *which goes over a pulley at* A, *and is fixed to the other end* D.

Draw AF, CK parallel to the horizon, FDE perp. to it, and KD perp. to CD; and let G be the center of gravity, w = weight of the beam. Then if the beam was to lie horizontally (Prop. XXXI. Cor. 2.) it would be GC : GD : : pressure at D : pressure at C; and GC : CD : : pressure at D : w; whence the pressure at D $= \frac{GC}{CD} w$, horizontally. And (Prop. XIV.) CD : CE : : $\frac{GC}{CD} w : \frac{CE \times CG}{CD^2} w$ = pressure in direction DK. Produce AD to O, and draw OI parallel to CD. Then the beam is sustained by three forces in the directions OD, DI and IO; and DI : DO : : S.IOD or ODC or ADC : rad; whence S.ADC : rad. : : $\frac{CE \times CG}{CD^2} \times w$: force DO or Q $= \frac{\text{rad.} \times CE \times CG}{\text{S.ADC} \times CD^2} w$. Therefore w : Q : : S.ADC : rad. $\times \frac{CE \times CG}{CD^2}$: : S.ADC : $\frac{CG}{CD} \times$ S.FDC, because $\frac{\text{rad.} \times CE}{CD} =$ S.EDC or FDC.

PROP. XXXIX. *Prob.*

21. *Let the beam* ED, *be sustained by the weights* P, Q, *by means of the ropes* DCP, EAQ, *going over the pullies* C, A, *in the horizontal line* AC. *To find the position of the beam; having the weights* P, Q, *given.*

One Way.

Let G be the center of gravity of the beam. Through D, E, draw HDS, FER perpendicular to AC.

AC. Then (Cor. 2. Prop. VIII.) S.EDS : P the tenſion of the thread CD : : S.CDS or CDH : $\frac{S.CDH}{S.EDS} \times P$ the tenſion of DE. Alſo, S.DER or EDS : Q : : S.AER or AEF : $\frac{S.AEF}{S.EDS} \times Q$ the tenſion of DE in a contrary direction. Then as the beam is in equilibrio, theſe forces or tenſions balance one another; therefore $\frac{S.CDH}{S.EDS} \times P = \frac{S.AEF}{S.EDS} \times Q$. Then P : Q : : S.AEF : S.CDH; which may be otherwiſe expreſſed, for AE : rad : : AF : S.AEF $= \frac{AF}{AE} \times$ rad; and DC : rad : : HC : $\frac{HC}{DC} \times$ rad = S.HDC. Whence P : Q : : $\frac{AF}{AE} : \frac{CH}{CD} : : \frac{AF}{HC} : \frac{AE}{DC}$.

Fig. 21.

Second Way.

Let R, S, be the perpendicular preſſures of the ends E, D; w = weight of the beam. Then (Cor. 2. Prop. XXXI.) R $= \frac{DG}{ED} w$, and S $= \frac{EG}{ED} w$. And (Cor. 2. Prop. VIII.) S.AED : R or $\frac{DG}{ED} w$: : S.AEF : tenſion of DE $= \frac{S.AEF \times DG}{S.AED \times ED} w$. And S.CDE : S or $\frac{EG}{ED} w$: : S.CDH : contrary tenſion of DE $= \frac{S.CDH \times EG}{S.CDE \times ED} w$, and theſe two forces of DE being equal, we have $\frac{S.AEF \times DG}{S.AED \times ED} w =$ S.CDH

Fig. 21. $\frac{S.CDH \times EG}{S.CDE \times ED} w$, and $\frac{S.AEF \times DG}{S.AED} = \frac{S.CDH \times EG}{S.CDE}$.

Whence $EG : DG :: \frac{S.AEF}{S.AED} : \frac{S.CDH}{S.CDE} :: S.AEF \times S.CDE : S.CDH \times S.AED$.

Third Way.

$S.CDE : S.EDS :: S : P = \frac{S.EDS}{S.CDE} \times S$, and $S.AED : S.RED :: R : \frac{S.RED}{S.AED} R = Q$. Then $P : Q :: \frac{S.EDS}{S.CDE} \times S : \frac{S.RED}{S.AED} \times R :: S.AED \times S : S.CDE \times R ::$ (laſt method) $S.AED \times \frac{EG}{ED} w : S.CDE \times \frac{DG}{ED} w :: S.AED \times EG : S.CDE \times DG$. And $S.AED : S.CDE :: P \times DG : Q \times EG$.

Fourth Way.

Draw Cm, Fn parallel to DE, and FE, HD perp. to the horizon. Then by the reſolution of forces, $CD : Dm :: P : \frac{Dm}{CD} P$ = perpendicular force at D; and $nE : EF :: Q : \frac{EF}{nE} Q$ = perpendicular force at E. Therefore $EG : GD :: \frac{Dm}{CD} P : \frac{EF}{nE} Q :: \frac{S.CDE}{S.CmD} \times P : \frac{S.FnE}{S.nFE} \times Q :: S.CDE \times P : S.AED \times Q$. For $S.CmD = S.mDF. = S.FED = S.nFE$. That is, $EG : GD :: S.CDE \times P : S.AED \times Q$. As in the third way.

Fifth Way.

Let R, S be the perpendicular weights of the ends E, D; or which is the ſame, the tenſions of the perpendicular

perpendicular ropes, FE, HD. By the reſolution of forces, if Cm, Fn be parallel to DE. The force FE or R is equivalent to En, Fn; and the force Dm or S, to DC, mC; therefore EF : Fn : : R : $\frac{Fn}{FE}$ R = force at E in direction Fn. And Dm : mC : : S : $\frac{mC}{mD}$ S = force at D in direction mC. But the beam being in equilibrio, theſe two oppoſite forces muſt be equal; therefore $\frac{Fn}{FE}$ R = $\frac{mC}{mD}$ S. Whence R : S : : $\frac{mC}{mD}$: $\frac{Fn}{FE}$: : $\frac{S.CDH}{S.CDE}$: $\frac{S.AEF}{S.AED}$: : S.AED × S.CDH : S.AEF × S.CDE. But (Cor. 2. Prop. XXXI.) R : S : : DG : EG. Whence DG : EG : : S.AED × S.CDH : S.AEF × S.CDE; the ſame as by the 2d method. And the ſame thing likewiſe follows from the 1ſt and 4th method together. Fig.

From theſe ſeveral ways of proceeding, it is evident, that which ever way we take, the proceſs if rightly managed always brings us to the ſame concluſion; and it comes to the ſame thing which way we uſe, ſo that we proceed in a proper manner. And this among other things, ſhews the great uſe and extent of that noble theory of the compoſition and reſolution of forces.

What is calculated above is concerning the angles, or the poſition of the ſeveral lines to one another, depending on the ſeveral forces. Then in regard to the weight of the beam, put it = w, then DC : Dm : : P : $\frac{Dm}{DC}$ P = S, and En : EF : : Q : $\frac{EF}{En}$ Q = R. And w = R + S = $\frac{Dm}{DC}$ P + $\frac{EF}{En}$ Q, an equation ſhewing the relation of the weights to one another.

Sixth

Fig. 21.

Sixth Way, by the Center of Gravity.

Produce AE, CD to B, and from B draw BGO through the center of gravity; which (by Prop. XXXVII.) will be perp. to AC, and therefore parallel to EF, DH. Then the angle EBG = AEF, and DBG = CDH. Then EB : BD : : S.BDE or CDE : S.BED or AED, and (Trigonom. B. II. Prop. V. Cor. 1.) EG : GD : : EB × S.EBG : BD × S.DBG : : EB × S.AEF : BD × S.CDH : : S.CDE × S.AEF : S.AED × S.CDH; the ſame as by the 2d way. Whence all the reſt will be had as before.

Cor. *It will be exactly the ſame thing, whether the weights* P, Q, *remain, or the ſtrings* AE, CD, *be fixed in that poſition to two tacks, any way in theſe lines. And if a beam* ED, *hang upon two tacks* A, C, *by ropes fixed there; it makes no difference, if you put two pullies inſtead of the tacks, for the ropes to go over, and then hang on weights* Q, P, *equal to the tenſions of the ſtrings* AE, CD.

For in both caſes, the forces or the tenſions of the ſtrings, and their directions, remain the ſame. And there is nothing elſe to make a difference in the ſituation of the beam.

Scholium.

Every one that knows any thing of mechanical principles will eaſily underſtand, that if any forces, which keep a body at reſt, be reſolved into others, to have the ſame effect; the contrary forces, or thoſe directly oppoſite, muſt act againſt a ſingle point; or elſe the equilibrium will be deſtroyed. And therefore in the preſent Prop. ſuppoſe any one ſhould divide the forces CD, AE, into the two HD, DY, and FE, EX, one perpendicular, the other parallel to the horizon. The forces HD, EF, will indeed balance the force of gravity at D and E, to which

which they are directly oppoſite. And therefore Fig.
the beam will remain unmoved by theſe. But the 21.
equal forces DY, EX, being parallel, never meet in a point; but acting obliquely on the beam, one of them drawing at D in direction DY, and the other at E in direction EX, the effect will be, that they will turn the beam ED about the center of gravity G. Therefore to prevent this, the forces DY, EX, muſt be ſubdivided; that is, they muſt be reſolved into others, one whereof is perp. to the horizon, the other parallel to ED. Then gravity will balance theſe perp. to the horizon, and the others, being equal and oppoſite, acting in the line EGD, act equally againſt any of the points D, G, or E. And ſo the beam will remain at reſt. But this is much better done at once at the firſt, by dividing DC, EF, each into two forces, one perp. to the horizon, the other parallel to the beam ED. And then the oppoſite forces will exactly balance one another, and the beam remain unmoved.

PROP. XL. *Prob.*

To find the poſition of the beam ED, *hanging by* 22.
the rope EBD, *whoſe ends are faſtened at* E *and* D, *and goes over a pulley fixed at* B.

Let G be the center of gravity of the beam, then (Prop. XXXVII.) BG will be perp. to the horizon. Then as the cord runs freely about the pulley B; therefore (Ax. 13.) the tenſion of the parts of the rope EB, BD are equal to one another, ſuppoſe = T. By the reſolution of forces, the force EB is equivalent to EG, GB; and DB to DG, GB. Therefore BE : EG :: T : $\frac{EG}{EB}$ T = force in direction EG. And BD : DG :: T : $\frac{DG}{BD}$ T = force in direction

Fig. 22. direction DG, which is equal and oppoſite to that in EG; therefore $\frac{EG}{EB}T = \frac{DG}{DB}T$. Whence EG : EB : : DG : DB. And therefore BG biſects the angle EBD.

Cor. *Hence* ED : *ſtring* EBD : : EG : EB *the part* EB *of the ſtring* : : *and ſo* GD : DB *the part* DB *of the ſtring.*

Scholium.

If GD be leſs than GE, then the center of gravity G, will be loweſt, when the beam hangs perpendicular with the end D downward. And in many caſes it will be higheſt, when it hangs perpendicular, with the end E downward.

PROP. XLI. *Prob.*

23. *There is a beam* BC *hanging by a pin at* C, *and lying upon the wall* BE; *to find the forces or preſſures at the points* B, *and* C, *and their directions.*

Produce BC to K, ſo that CK may be equal to CB. Draw BA parallel, and CL perpendicular, to the horizon; and draw BL, CN, KI perp. to BCK. Through the center of gravity G, draw GF parallel to CL. By Prop. XIV. if a body lies upon an inclined plane, as BC; its weight, its inclination down the plane, and preſſure againſt it, are as BC, CA and AB; that is, as CL, CB and BL. Therefore if CL repreſent the weight of the body, CB will be the force of urging it down the plane, and BL the total preſſure againſt the plane. And ſince GF is parallel to CL, BL is divided in F, in the ſame ratio, as BC is divided in G. And therefore (Cor. 2. Prop. XXXI.) BF will be the part of the preſſure acting at C, and FL the part acting at B. Make CN equal to BF, and compleat the

the parallelogram CNIK, and draw CI. Then since BC or CK is the force in direction CK, and CN the force in direction CN; then by compoſition, CI will be the ſingle force by which C is ſuſtained, and CI its direction. But the triangles CKI, CBF are ſimilar and equal, and CI = CF, and in the ſame right line; therefore CF is the quantity and direction of the force acting at C to ſuſtain it. Therefore the weight of the body, the preſſure at B, and the force at C; are reſpectively as CL, FL, and CF. Fig. 23.

Cor. 1. *Produce* FG *to interſect* CN *in* H; *then the weight of the body, the preſſure at* B, *and the force acting at* C; *are reſpectively as* HF, HC, *and* CF.

For in the parallelogram CLFH, HF = CL, and HC = FL.

Cor. 2. *If the beam was ſupported by a pin at* B, *and laid upon the wall* AC; *the like conſtruction muſt be made at* B, *as has been done at* C, *and then the forces and directions will be had.*

Cor. 3. *If a line perp. to the horizon be drawn through* F, *where the direction of the forces* CF, *and* BF *meet; it paſſes through* G *the center of gravity of the beam.*

Cor. 4. *It is all one whether the beam is ſuſtained by the pin* C *and the wall* BE, *or by two ropes* CI, BP *acting in the directions* FC, FB, *and with the forces* CF, FL.

Scholium.

The proportions and directions of the forces here found, are the ſame as in Prop. LXIV. of my large book of Mechanics, and obtained here by a different method. The principles here uſed are indiſputable; and the principle made uſe of in that

LXIV.

Fig. 23. LXIV. Prop. is here demonſtrated in the third Cor. ſo that the reader may depend upon the truth of them all.

PROP. XLII. *Prob.*

24. BC *is a heavy beam ſupported upon two poſts* KB, LC; *to find the poſition of the poſts, that the beam may reſt in equilibrio.*

Let G be the center of gravity; draw BA parallel to the horizon, and BF, GD, CAN perpendicular to it. Then (Prop. XXXI. Cor. 2.) if BC be the weight of the body, CG will be the part of the weight acting at B, and BG the weight at C. Therefore make CN = BG, and BF = GC; and from N and F, draw NI, FK, parallel to BC; and make NI = FK, of any length, and lying contrary ways. Then draw IC and KB, which will be the poſition of the poſts required.

For BF is the weight upon B; and CN, that upon C, which forces being in direction of the lines BF, CN, the beam will remain at reſt by theſe forces. And the forces NI, FK, in direction BC, being equal and contrary, will alſo keep the beam in equilibrio, therefore the forces KB, IC, compounded of the others, will alſo keep the beam in equilibrio, acting in the directions KB, IC, or MB, LC.

Cor. 1. *Hence if* DG *be produced, it will paſs through the interſection* H, *of the lines* LC, MB.

For the triangles INC, CGH are ſimilar; therefore IN : NC : : CG : GH, the interſection with CL. Alſo the triangles KFB, BGH are ſimilar; therefore KF : BG : : BF : GH the interſection with MB, which muſt needs be the ſame as the other, ſince the three firſt terms of the proportion are the ſame; for KF = NI, BG = NC, and BF = CG.

Cor.

Cor. 2. *If a line be drawn through the center of gravity* G, *of the beam, perpendicular to the horizon; and from any point* H *in that line, (above or below* G) *the lines* HBK *and* HCM *be drawn; then the props* BM *and* CL *will ſuſtain the beam in equilibrio.* Fig. 24.

Cor. 3. *If* GO *be drawn parallel to* HC; *then the weight of the beam, the preſſure at* C, *and thruſt or preſſure at* B; *are reſpectively as* HG, OG, *and* HO, *and in theſe directions.*

Cor. 4. *It is all one for maintaining the equilibrium, whether the beam* BC *be ſupported by two poſts or props* MB, LC; *or by two ropes* BH, CH; *or by two planes perpendicular to theſe ropes at* B *and* C.

For in all theſe caſes the forces and directions are the ſame; and there is nothing elſe concerned, but the forces and directions.

Scholium.

It does not always happen that the center of gravity is at the loweſt place it can get, to make an equilibrium. For here if the beam BC be ſupported by the poſts MB, LC; the center of gravity is at the higheſt it can get; and being in that poſition, it has no inclination to move one way more than another, and therefore it is as truly in equilibrio, as if it was at the loweſt point. It is true, any the leaſt force will deſtroy that equilibrium, and then if the beam and poſts be movable about the angles M, B, C, L, which is all along ſuppoſed, the beam will deſcend till it is below the points M, L, and gain ſuch a poſition as deſcribed in Prop. XXXIX. and its Cor. ſuppoſing the ropes fixed at A, C (fig. 21.); and then G will be at the loweſt point, and will come to an equilibrium again. In planes the center of gravity G may be either at its higheſt or loweſt point. And there are caſes, when the

Fig. 24. the center of gravity is neither at its higheſt nor loweſt, as may happen in the caſe of Prop. XL. ſo that they who who contend, that in caſe of an equilibrium, the center of gravity muſt *always* be at the loweſt place, are much miſtaken, and know little about the principles of mechanics, or the nature of things.

Thoſe that want to ſee more variety about the motion of bodies, on inclined planes; the preſſure, and direction of the preſſure of beams of timber; centers of gravity, and alſo the centers of oſcillation and percuſſion, &c. may conſult my large book of Mechanics.

SECT.

Fig.

SECT. IV.

The MECHANIC POWERS; *the Strength and Streſs of Timber.*

PROP. XLIII.

IN a balance, where the arms are of equal length; if two equal weights be ſuſpended at the ends, they will be in equilibrio.

The balance is a ſtraight inflexible rod or beam, turning about a fixed point or axle in the middle of it; to be loaded at each end with weights ſuſpended there. 25.

Let AB be the beam or lever, C the middle point or center of motion; D, E the weights hanging at the ends A and B. Then let the beam and the weights, or the whole machine, be ſuſpended at C. And ſuppoſe the beam and the weights be turned about upon the center C; then the points A, B being equidiſtant from C will deſcribe equal arches, and therefore the velocities will be equal, and if the bodies D and E be equal, then the motion of D will be equal to the motion of E, as their quantities of matter and velocities are equal; and conſequently, if the beam and weights are ſet at reſt, neither of them can move the other, but they will remain in equilibrio.

Cor. *If one weight be greater than the other; that weight and ſcale will deſcend, and raiſe the other.*

SCHOLIUM.

The uſe of the balance, or a common pair of ſcales, is to compare the weights of different bo-

Fig. 25. dies; for any body whoſe weight is required, being put into one ſcale, and balanced by known weights put into the other ſcale, theſe weights will ſhew the weight of the body. To have a pair of ſcales perfect, they muſt have theſe properties. 1. The points of ſuſpenſion of the ſcales, and the center of motion of the beam, A, C, B, muſt be in a right line. 2. The arms AC, BC, muſt be of equal length from the center. 3. That the center of gravity be in the center of motion C. 4. That there be as little friction as poſſible. 5. That they be in equilibrio, when empty.

If the center of gravity of the beam be above the center of motion, and the ſcales be in equilibrio, if they be put a little out of that poſition, by putting down one end of the beam, that end will continually deſcend, till the motion of the beam is ſtopt by the handle H. For by that motion, the center of gravity is continually deſcending, according to the nature of it. But if the center of gravity of the beam be below the center of motion; if one end of the beam be put down a little, to deſtroy the equilibrium; it will return back and vibrate up and down. For by this motion the center of gravity is endeavouring to deſcend.

PROP. XLIV. *Prob.*

25. *To make a falſe balance; or one which is in equilibrio when empty, and alſo in equilibrio, when loaded with unequal weights.*

Make ſuch a balance as deſcribed in the laſt Prop. only inſtead of making the center of motion in the middle at C, make it nearer one end, as at F. And make the weight of the ſcales ſuch that they may be in equilibrio upon the center F. Then if two weights D, E, be made to be in equilibrio in the two ſcales; theſe weights will be unequal, for

for they will be reciprocally as the lengths of the arms AF, BF. That is, AF : BF : : E : D. Fig. 25.

For (Prop. XXXI. Cor. 1.) ſince F is the center of gravity of D and E, ſuppoſing them to act at A and B; therefore FA × D = FB × E. And FA : FB : : E : D. But AF is greater than FB, therefore E is greater than D.

Cor. 1. *Hence to diſcover a falſe balance, make the weights in the two ſcales to be in equilibrio; then change the weights to the contrary ſcales. And if they be not in equilibrio, the balance is falſe.*

Cor. 2. *Hence alſo to prove a pair of good ſcales, they muſt be in equilibrio when empty, and likewiſe in equilibrio with two weights. Then if the weights be changed to the contrary ſcales, the equilibrium will ſtill remain, if the ſcales are good.*

Cor. 3. *From hence alſo may be known what is gained or loſt, by changing the weights, in a falſe balance.*

Take any weight, as 1 pound, to be put into one ſcale and balanced by any ſort of goods in the other. Since AF × D = BF × E; let the weight D be 1; then $E = \frac{AF}{BF}$ the weight of the goods in the ſcale E. Then changing the ſcales, let the weight E be 1; then $D = \frac{BF}{AF}$ the weight of the goods in the ſcale D. Then $\frac{AF}{BF} + \frac{BF}{AF}$ = whole weight of the goods, and $\frac{AF}{BF} + \frac{BF}{AF} - 2$ = gain to the buyer in 2lb. &c. Therefore $\frac{AF^2 + BF^2 - 2AF \times BF}{AF \times BF}$ = gain in 2 lb. = $\frac{\overline{AF - BF}^2}{AF \times BF}$; and $\frac{\overline{AF - BF}^2}{2AF \times BF}$ = gain in 1lb.

Fig. 25. Therefore if w is any weight to be bought; the gain to the buyer, for the weight w, by changing the ſcales, will be $\frac{\overline{AF-BF}^2}{2AF \times BF} w$. For example, if AF be 16, and BF 15; then the gain will be $\frac{\overline{16-15}^2}{2 \times 16 \times 15} w = \frac{1}{480} w$.

Scholium.

In demonſtrating the properties of the mechanical powers; ſince the weight is commonly ſome large body whoſe weight is to be overcome or balanced; therefore the power which acts againſt it, will be moſt fitly repreſented by another weight; and thus the power and weight being of the ſame kind, may moſt properly and naturally be compared together. For ſuch a weight may repreſent any power, as the ſtrength of a man's hand, the force of water or wind, &c. And this weight repreſenting the power, being ſuſpended by a rope, may hang perpendicular where the power is to act perpendicular to the horizon; or may go over a pulley when it acts obliquely.

PROP. XLV.

26. *If the power and weight be in equilibrio upon any*
27. *lever, and act in lines perpendicular to the lever:*
28. *then the power* P *is to the weight* W; *as the diſtance*
29. *of the weight from the ſupport* C, *is to the diſtance of the power from the ſupport.*

There are four ſorts of levers. 1. When the ſupport is between the weight and the power. 2. When the weight is between the power and the ſupport. 3. When the power is between the weight and the ſupport. 4. When the lever is not ſtraight but crooked.

A lever

A lever is any inflexible rod or beam, of wood or metal, made uſe of to move bodies. The ſupport is the prop it reſts on, in moving or ſuſtaining any heavy body, and this is the ſame as the center of motion. Fig. 26. 27. 28. 29.

Let the power P act at A by means of a rope; then as C is the prop or center of motion, if the lever be made to move about the center C, the arches deſcribed by A and W, that is, the velocities of A and W, will be as the radii CA and CW. But the velocity of P is the ſame as that of the point A. Therefore velocity of P : velocity of W : : CA : CW : : (by ſuppoſition) W : P; therefore P × velocity of P = W × velocity of W. Conſequently their motions are equal, and they cannot move one another, but muſt remain in equilibrio. And if they be in equilibrio, they muſt have this proportion aſſigned.

Cor. 1. *If a power* P *act obliquely againſt the lever* WA; *the power and weight will be in equilibrio, when the power* P *is to the weight* W, *as the diſtance of the weight* CW, *to* CD *the perpendicular, drawn from the ſupport to the line of direction of the power.* 30.

For in this caſe WCD becomes a bended lever, and the power P acts perpendicular to CD at D; and (Ax. 12.) it is all one whether the power acts at D or A, in the line of direction AD. And hence

Cor. 2. *If any force be applied to a lever* ACW *at* A, *its power to turn it about the center of motion* C, *is as the ſine of the angle of application* CAD.

For if CA be given, CD is as the ſine of CAD.

Cor. 3. *In a ſtraight lever, of theſe three, the power, the weight, and the preſſure upon the ſupport; the middlemoſt is equal to the ſum of the other two.*

For the middle one acts againſt both the others, and ſupports them.

Fig. 30. Cor. 4. *From the foregoing properties of the lever, the effects of ſeveral ſorts of ſimple machines may be explained; and likewiſe the manner of lifting, carrying, drawing of heavy bodies, and ſuch like.*

26. For example, if a given weight W is to be raiſed by a ſmall power applied at A, the end of the lever AW. If we divide WA in C, ſo that it be as CA : CW : : as the weight W : to the power P; then fixing a prop or ſupport at C, or rather a little nearer W; then the power P with a ſmall addition, will raiſe the weight W.

27. Again, if two men be to carry a weight W, upon the lever CA. The weight the man at A carries, is to the weight the man at C carries, as CW to AW. And therefore the lever or beam CA ought to be divided in that proportion at W, the place where the weight is to lie, according to the ſtrength of the men that carry it.

31. Likewiſe if two horſes be to draw at the ſwingtree AB, by the ropes AF, BG; and the ſwingtree draws a carriage, &c. by the rope CD; then the force acting at A will be to the force acting at B, as BC to AC. And therefore BC ought to be to AC, as the ſtrength of the horſe at F, to the ſtrength of the horſe at G; the weaker horſe having the longer end. But it is proper to make the croſs bar AB crooked at C; that when the ſtronger horſe pulls his end more forward, he may pull obliquely, and at a leſs diſtance from the center; whilſt the weaker horſe pulls at right angles to his end, and at a greater diſtance.

Again, ſuch inſtruments as crows and handſpikes are levers of the firſt kind, and are very uſeful and handy for raiſing a great weight to a ſmall height. Alſo ſciſſars, pinchers, ſnuffers, are double levers of the firſt kind, where the joint is the fulcrum or ſupport. The oars of a boat, the rudder of a ſhip, cutting knives fixed at one end, are levers of the

ſecond

ſecond kind. Tongs, ſheers, and the bones of animals, are levers of the third kind. A ladder raiſed upright, is a lever of the third kind. A hammer drawing out a nail is a lever of the fourth kind. Fig. 32.

The *Steel Yard* is nothing but a lever of the firſt kind, whoſe mechaniſm or conſtruction is this. Let AB be the beam, C the point of ſuſpenſion; P the power, movable along the beam CB. The beam being ſuſpended at D, move the power P, along towards C, till you find the point O, where P reduces the beam to an equilibrium. Then at A hang on the weight W of 1 pound; and move P to be in equilibrio with it at 1; then hang on W of 2 pound, and ſhift P till it be in equilibrio, at 2. Proceed thus with 3, 4, 5, &c. pounds at W, and find the diviſions 3, 4, 5, &c. Or, if you will, after having found the points O, 1; make the diviſions, 12, 23, 34, &c. each equal to O1. But for more exactneſs and expedition, having found the point O, where P makes the beam in equilibrio: hang on any known number of pounds, as W; and move P to the point B, where it makes an equilibrium. Then divide OB into as many equal parts as W conſiſts of pounds: mark theſe diviſions 1, 2, 3, 4, &c. Then any weight W being ſuſpended at A; if P be placed to make an equilibrium therewith, then the number where P hangs will ſhew the pounds or weight of W.

To prove this, we muſt obſerve, that AC is the heavier end of the beam; therefore let Q be the *Momentum* at that end to make an equilibrium with P ſuſpended at O; that is, let $Q = CO \times P$. But (Cor. 2. Prop. XXXVI.) $Q + CA \times W = CF \times P = CO \times P + OF \times P$. Take away $Q = CO \times P$, and then $CA \times W = OF \times P$. Whence AC : P : : OF : W. But AC and P are always the ſame; therefore W is as OF; that is,

Fig. 32. if OF be 1, 2, 3, &c. diviſions, then W is 1, 2, 3, &c. pounds.

We may take notice that the diviſions O1, 12, 23, &c. are all equal; but CO may be greater or leſſer, or nothing.

If you would know how much the weight P is, take the diſtance CA, and ſet it from O along the diviſions O, 1, 2, 3, &c. and it will reach to the number of pounds. But this is of no conſequence, being only matter of curioſity.

PROP. XLVI.

33. *In the compound lever, or where ſeveral levers act upon one another, as* AB, BC, CD, *whoſe ſupports are* F, G, I; *the power* P : *is to the weight* W : : *as* BF × CG × DI : *to* AF × BG × CI.

For the power P acting at A : force at B : : BF : AF; and force or power at B : force at C : : CG : GB; and force or power at C : weight W : : DI : IC. Therefore *ex æquo*, power P : weight W : : BF × CG × DI : AF × GB × IC.

And it is the ſame thing in the other kinds of levers, taking the reſpective diſtances, from the ſeveral props or ſupports.

PROP. XLVII.

34. *In the wheel and axle; the weight and power will be in equilibrio, when the power* P *is to the weight* W; *as the radius of the axle* CA, *where the weight hangs; to the radius of the wheel* CB, *where the power acts.*

This is a wheel fixed to a cylindrical roller, turning round upon a ſmall axis; and having a rope going round it.

Through

Through the center of the wheel C, draw the horizontal line BCA. Then BP and AW are perpendicular to BA; and BCA will be a lever whose support is C; and the power acts always at the distance BC, and the weight at the distance CA; which remain always the same. Therefore the weight and power acts always upon the lever BCA. But by the property of the lever (Prop. XLV.) BC : CA : : W : P, to have an equilibrium. Fig. 34.

Otherwise.

If the wheel be set a moving, the velocity of the point A or of W, is to that of B or P, as CA to CB; that is (by supposition) as P : W. Therefore W × velocity of W = P × velocity of P; therefore, the motions of P and W being equal, they cannot, when at rest, move one another.

Cor. 1. *If the power acting at the radius* CB, *act not at right angles to it; draw* CD *perpendicular to* BP *the direction of the power; then the power* P : *is to the weight* W : : *as the radius of the axle* CA : *to the perpendicular* CD. 35.

For in the lever DCA, whose support is C, the power P : weight W : : CA : CD.

Cor. 2. *In a roller turning round on the axis or spindle* FC, *by the handle* CBG; *the power applied perpendicularly to* BC *at* B, *is to the weight* W : : *as the radius of the roller* DA, *to the length of the handle* CB. 36.

For in turning round, the point B describes the circumference of a circle; the same as if it was a wheel whose radius is CB.

SCHOLIUM.

All this is upon supposition that the rope sustaining the weight is of no sensible thickness. But if it is a thick rope, or if there be several folds of it

Fig. 36. it about the roller or barrel; you muſt meaſure to the middle of the outſide rope to get the radius of the roller. For the diſtance of the weight from the center is encreaſed ſo much, by the rope's going round.

From hence the effects of ſeveral ſorts of machines, or inſtruments may be accounted for. A roller and handle for a well or a mine, is the ſame thing as a wheel and axle, a windlaſs and a capſtain in a ſhip is the ſame; and ſo is a crane to draw up goods with. A gimblet and an augur to bore with, may be referred to the wheel and axle.

The wheel and axle has a particular advantage over the lever; for a weight can but be raiſed a very little way by the lever. But by continual turning round of the wheel and roller, the weight may be raiſed to any height required.

PROP. XLVIII.

37. *In a combination of wheels with teeth; if the power* P *be to the weight* W : : *as the product of the diameters of all the axles or pinions, to the product of the diameters of all the wheels; the power and weight will be in equilibrio.*

AC, CD are the radii of one wheel and its axle; DG, GH, the radii of another; and HI, IK are thoſe of another. Theſe act upon one another at D and H, then as the power or force P is propagated through all the wheels and axles to W; we muſt proceed to find the ſeveral forces acting upon them, by Prop. XXXVII. Thus,

$$CD : CA : : P : \frac{CA}{CD} P = \text{force acting at D.}$$

$$\text{and } GH : GD : : \frac{CA}{CD} P \text{ (force at D)} : \frac{CA \times GD}{CD \times GH} P$$

$$= \text{force acting at H.}$$

and

and IK : IH : : $\frac{CA \times GD}{CD \times GH}$ P (force at H) : $\frac{CA \times GD \times IH}{CD \times GH \times IK}$ P = force at K = W. And CA × GD × IH × P = CD × GH × IK × W; whence P : W : : CD × GH × IK : : CA × GD × IH. Fig. 37.

Cor. 1. *If the weight and power be in equilibrio, and made to move; the velocity of the weight is to the velocity of the power; as the product of the diameters of all the axles or pinions, to the product of the diameters of all the wheels. Or inſtead of the diameters, take the number of teeth in theſe axles and wheels that drive one another. And the ſame is true of wheels carried about by ropes.*

For the power is to the weight; as the velocity of the weight to the velocity of the power. And the number of teeth in the wheels and pinions, that drive one another, are as the diameters. And the ropes ſupply the place of teeth.

Cor. 2. *In a combination of wheels with teeth. The number of revolutions of the firſt wheel, is to the number of revolutions of the laſt wheel, in any time; as the product of the diameters of the pinions or axles, to the product of the diameters of the wheels: or as the product of the number of teeth in the pinions, to the product of the number of teeth in the wheels which drive them. And the ſame is true of wheels going by cords.*

For as often as the number of teeth in any pinion is contained in the number of teeth of the wheel that drives it; ſo many revolutions does that pinion make for one revolution of the wheel.

SCHOLIUM.

A pinion is nothing but a ſmall wheel, fixed at the other end of the axis, oppoſite to the wheel; and

Fig. 37. and conſiſts but of a few leaves or teeth; and therefore is commonly leſs than the wheel. But in the ſenſe of this propoſition, a pinion may, if we pleaſe, be bigger than the wheel. As if we put the power and weight into the contrary places, the wheels will become the pinions, and the pinions the wheels, according to the meaning of this propoſition.

PROP. XLIX.

If a power ſuſtains a weight by means of a fixed pulley; the power and weight are equal: but if the pulley be movable along with the weight, then the weight is double the power.

A pulley is a ſmall wheel of wood or metal, turning round upon an axis, fixed in a block; on the edge of the pulley is a groove for the rope to go over.

38. 39. Through the centers of the pullies, draw the horizontal lines AB, CD; then will AB repreſent a lever of the firſt kind, and its ſupport is the center of the pulley, which is a fixed point, the block being fixed at F. And the points A, and B, where the power and weight act, being equally diſtant from the ſupport, therefore (Prop. XLV.) the power P = weight W.

Alſo CD repreſents a lever of the ſecond kind, whoſe ſupport is at C, a fixed point; the rope CG being fixed at G. And the weight W acting at the middle of CD, and the power acting at D, twice the diſtance from C; therefore (Prop. XLV.) the power P is to the weight W : : as $\frac{1}{2}$CD to CD; or as 1 to 2.

Cor. *Hence all fixed pullies are levers of the firſt kind, and ſerve only to change the direction of the motion; but make no addition at all to the power.*

And

And therefore if a rope goes over ſeveral fixed pullies; the power is not increaſed, but rather decreaſed, by the friction. Fig. 38. 39.

Scholium.

The uſe of a fixed pulley is of great ſervice in raiſing a weight to any height, which otherwiſe muſt be carried by ſtrength of men, which is often impracticable. Therefore if a rope is fixed to the weight at W (fig. 38.) and paſſed over the pulley BA; a man taking hold at P will draw up the weight, without moving from the place. And if the weight be large, ſeveral perſons may pull together at F, to raiſe the weight up; where in many caſes they cannot come to it, to raiſe it by ſtrength.

PROP. L.

In a combination of pullies, all drawn by one rope going over all the pullies; if the power P *is to the weight* W; *as* 1 *to the number of the parts of the rope proceeding from the movable block and pullies. Then the power and weight will be in equilibrio.* 40.

Let the rope go from the power about the pullies in this order, *ntovrs*, where the laſt part *s* is fixed to the lower block B. Now (Ax. 13.) all the parts of the rope *ntovrs* are equally ſtretched, and therefore each of them bears an equal weight; but the part *n* bears the power P, which goes to the fixed block A. All the other parts ſuſtain the weight and movable block B, each with a force equal to P. Therefore P is to the ſum of all the forces, ſuſtained by *o*, *r*, *s*, *t*, *v*, or the weight W, as 1 to the number of theſe ropes immediately communicating with the movable block B. And all the ropes having an equal tenſion, none of them

Fig. 40. them can move the reſt, but they muſt remain in equilibrio.

And if you take away the power at P, and apply a force at the rope *t* equal to P, to pull upwards in direction *t*A; this will make no alteration, for the rope *t* draws from the movable block with the ſame force as before, and therefore the weight is ſuſtained as before; for the upper pulley (by Prop. XLIX. Cor.) which the rope *nt* goes over, ſerves only to change the direction. And therefore as there are the ſame number of ropes ſtill drawing from the movable block as before; the propoſition holds good alſo in this reſpect. And it would be the ſame thing if the rope *s* was fixed to the weight W inſtead of the block B; but had it been fixed to the block A, there muſt have been a pulley more below, and a rope more, which would have increaſed the power, according to the propoſition.

Cor. 1. *Hence it appears to be a diſadvantage to the power to pull againſt the fixed block.*

For the rope *n* has no more purchaſe, or no more effect than the rope *t* has which draws againſt the movable block; and therefore when one draws by the rope *n*, there muſt be a pulley more, which will create more friction.

Cor. 2. *Hence one may explain the effects of all ſorts of machines compoſed of pullies; or find out ſuch a conſtruction or combination of them as to anſwer any purpoſe deſired. And to find its force, begin at the power and call it* 1; *then all the parts of the running rope that go and return about ſeveral pullies, muſt be each numbered alike. And any rope that acts againſt ſeveral others muſt be numbered with the ſum of theſe. And ſo on to the weight.*

For

For example; ſuppoſe a man wanted to draw him- Fig.
ſelf up to the top of a houſe or a church. Get a 41.
pulley A fixed at the top, and place another B at the bottom. Let a rope be fixed to the upper block A, and brought down about the pulley B, and then put round the upper pulley, and ſo brought to the ground at H. Then if a croſs ſtick CD be faſtened to the block B by a rope ; a man may get aſtride of the ſtick, and then draw himſelf up by the rope H. And the power to draw himſelf up, will be little more than $\frac{1}{3}$ of his weight. For the power at H, and the two parts of the rope going about the pulley B, ſuſtain all his weight ; and each of them ſuſtains one third of it.

If inſtead of the ſtick CD, he takes a chair to ſit in ; then when he has drawn himſelf up to any height he pleaſes, he may fix the rope H to the chair, and then do any ſort of buſineſs, as ſet up a dial, point the walls, and ſuch like, as is commonly done.

Again, ſeveral tackles are uſed aboard a ſhip, 42.
for hoiſting goods and the like. Let A, B be two blocks with pullies, the upper one being fixed, and let a weight W be ſuſpended at the ſingle pulley and rope, one end of the rope being fixed at F, and the other faſtened to the movable block B. This pulley and rope BCF is called a *Runner*. Let the power be at P, call it 1 ; then all the ropes going from B to A, muſt be each of them 1, and the rope going from the block B, acting againſt theſe four muſt be marked 4, and the other part of it CF muſt alſo be 4. Laſtly, the weight acting againſt theſe two , muſt be 8. And then the power P is to the weight W, as 1 to 8.

ABCD is another tackle with a runner BAD, A 43.
being a fixed pulley ; the two blocks B, C, are both movable. The rope DAB is fixed to the weight

Fig. 43. weight at D, and to the block B. The rope PB goes and returns about the pullies BC, and at laſt is faſtened to the block C. Let P be the power, mark it 1, then the other parts of the rope between the blocks, muſt alſo be 1 apiece. Then CI acting againſt 3, muſt be 3. And AB is 4, as it acts againſt 4; likewiſe AD muſt be 4. Therefore the whole force that ſuſtains the weight W is 3 and 4, or 7. And the power to the weight as 1 to 7.

44. The following is a ſort of *Spaniſh burton*, A and F are two fixed pullies; C and B two movable ones. The rope going from the power P, goes round C, B, and A, and is faſtened to the block B. Another rope is faſtened to the block B, and goes over the pulley F, and is fixed to the block C. Then marking the power P, 1. Then each part of the rope, continued over C, B, and A to B again muſt be each 1. Then FC muſt be 2, as it acts againſt two parts; and likewiſe the other part of it FB muſt be 2. Then the whole that lifts the weight W, is $1 + 1 + 1 + 2 = 5$. And therefore the power is to the weight as 1 to 5.

The friction between the pullies and blocks is ſometimes conſiderable. To remedy which, they muſt be as large as they can conveniently be made, and kept oiled or greaſed.

PROP. LI.

45. *In the ſcrew, if the power applied at* E, *be to the weight, preſſure, &c. at* B; *as the diſtance of two threads of the ſcrew, taken parallel to the axis of it, is to the circumference deſcribed by the power at* E; *then the weight and power will be in equilibrio.*

A ſcrew is an inſtrument conſiſting of two parts AB, CD, fitting into one another. AB is the male ſcrew, called the top or ſpindle; this is a long cylindrical body, having its ſurface cut into ridges and

and hollows, that run round it in a ſpiral manner from one end to the other, at equal diſtances; theſe riſings are called threads, and ſo many revolutions as they make, ſo many threads the ſcrew contains. CD is the female ſcrew, or the plate, through which the other goes; its concavity is cut in the ſame manner as the male, ſo that the ridges of the male may exactly fit the hollows of the female. By reaſon of the winding of the threads, as the handle EF is turned one way or the other, the male AB goes further in or comes further out, of the female. Fig. 45.

Let the point E of the handle, make one revolution, then the male AB will have advanced the diſtance between one thread and another, of the ſcrew. Therefore if G repreſent any weight, which the end B acts againſt, it will be moved through the breadth of a thread, whilſt the power moves through a circumference whoſe radius is EA. Therefore the velocity of G is to the velocity of E, as the breadth of a thread, to the circumference deſcribed by E; that is (by ſuppoſition) as the power at E, to the weight at G. Therefore E × velocity of E = G × velocity of G; and therefore their motions being equal, they will be in equilibrio.

Cor. *By reaſon of the friction, if any weight is to be removed by a ſcrew; the power muſt be to the weight; at leaſt as the breadth of two threads of the ſcrew, to the circumference deſcribed by the power; to keep the weight in equilibrio; and muſt be much more to move it.*

For in the ſcrew there is ſo much friction, that it will ſuſtain the weight when the power is taken away. And therefore the friction is as great or greater than the power. And therefore the whole power applied muſt at leaſt be doubled to produce any motion.

Fig. 45.

SCHOLIUM.

Screws with ſharp threads have far more friction than thoſe with ſquare threads, and therefore move a body with more difficulty.

The ſcrew, in moving a body acts like an inclined plane. For it is juſt the ſame as if an inclined plane was forced under a body to raiſe it; the body being prevented from flying back, and the baſe of the plane being driven parallel to the horizon.

The uſe of this power is very great. It is of great ſervice for fixing ſeveral things together by help of ſcrew nails; it is likewiſe very uſeful for ſqueezing or preſſing things cloſe together, or breaking them; alſo for raiſing or moving large bodies. The ſcrew is uſed in preſſes for wine, oil, or for ſqueezing the juice out of any fruit. The very friction of this machine has its particular uſe, for when a weight is raiſed to any height; if the power be taken away, the ſcrew will retain its poſition, and hinder the weight from deſcending again by its friction, without any other power to ſuſtain it.

In the common ſcrew, ſuch as is here ſuppoſed; the threads are all one continued ſpiral from one end to the other; but where there are two or more ſpirals, independent of one another, as in the worm of a jack; you muſt meaſure between thread and thread of the ſame ſpiral, in computing the power.

Fig.

PROP. LII.

In the endleſs ſcrew, where the teeth of the worm or ſpindle AB, *drives the wheel* CD, *by acting againſt the teeth of it. If the power applied at* P, *is to the weight* W, *acting upon the edge of the wheel at* C : : *as the diſtance of two threads or teeth, between fore ſide and fore ſide, taken along* AB ; *is to the circumference deſcribed by the power* P. *Then the weight and power will be in equilibrio.* 46.

The *endleſs* or *perpetual ſcrew* is one that turns perpetually round the axis AB ; and whoſe teeth fit exactly into the teeth of the wheel CD, which are cut obliquely to anſwer them : ſo that as AB turns round, its teeth take hold of the teeth of the wheel CD, and turns it about the axis I, and raiſes the weight W.

For by one revolution of the power at P, the wheel will be drawn forward one tooth ; and the weight W will be raiſed the ſame diſtance. Therefore the velocity of the power, will be to that of the weight; as that circumference, to one tooth : : that is (by ſuppoſition) as the weight W, to the power P. Therefore the power P × velocity of P = W × velocity of W ; therefore their motions being equal, they will be in equilibrio.

Cor. *If a weight* N *be ſuſpended at* E *on the axle* EF ; *then if the power* P, *is to the weight* N : : *as the breadth of a tooth* × EF, *to the circumference deſcribed by* P × CD. *They will be in equilibrio.*

Or if the power P ; *is to the weight* N : : *as radius of the axle* EI, *to the radius of the handle* FP × *by the number of teeth in* CD ; *they will be in equilibrio.*

Fig. 46. For power P : weight W :: 1 tooth : circumference. and weight W : weight N : : EF : CD. therefore P : N :: 1 tooth × EF : CD × circumference.

Or thus, whilſt EF turns round once, P turns round as oft as CD has teeth; whence EI : BP × number of teeth : : velocity of N : velocity of P : : P : N.

Scholium.

47. As the teeth of the wheel CD, muſt be cut obliquely to anſwer the teeth or ſcrew on AB; ſuppoſing AB to lie in the plane of the wheel CD; and therefore the wheel will be acted on obliquely by the ſcrew AB. To remedy that, the ſcrew AB may be placed oblique to the wheel, in ſuch a poſition, that when the teeth of the wheel are cut ſtraight or perp. to its plane, the teeth of the ſcrew AB, may coincide with them, and fit them. By that means the force will be directed along the plane of the wheel CD. Fig. 47 explains my meaning.

This machine is of excellent uſe, not only in itſelf, for raiſing great weights, and other purpoſes; but in the conſtruction of ſeveral ſorts of compound engines.

PROP. LIII.

48. *In the wedge* ACD, *if a power acting perpendicular to the back* CD, *is to the force acting againſt either ſide* AC, *in a direction perpendicular to it; as the back* CD, *to either of the ſides* AC; *the wedge will be in equilibrio.*

A wedge is a body of iron or ſome hard ſubſtance, in form of a priſm, contained between two iſoceles triangles, as CAD. AB is the height, and CD the back of it; AC, AD the ſides.

Let AB be perp. to the back CD, and BE, BF, perp. to the ſides AC, AD. Draw EG, FG parallel

rallel to BF, BE; then all the ſides of the paral- Fig.
lelogram BEGF are equal. The triangles EGB, 48.
ADC are ſimilar; for draw EOF, which will be perp. to AB; then the right angled triangles AEB, AEO, are ſimilar, and the angle ABE = AEO = ACB; that is, GBE = ACD, and likewiſe BGE = ADC, whence CAD = BEG.

Now let BG be the force acting at B, in direction BA, perp. to CD; then (Prop. IX.) the forces againſt the ſides AC, AD, will be in the directions EB, FB; and therefore EB, EG will repreſent theſe forces (by Prop. VIII.), when they keep one another in equilibrio. Therefore force BG applied to the back of the wedge, is to the force BE, perp. to the ſide AC; as BG to BE; that is, (by ſimilar triangles) as CD to CA.

Cor. 1. *The power acting againſt the back at* B, *is to that part of the force againſt* AC, *which acts parallel to the back* CD; *as the back* CD, *is to the height* AB.

For divide the whole force BE into the two BO, OE; the part EO acts parallel to CD; therefore the force acting at B, is to the force in direction OE or BC; as BG to OE; that is, (by ſimilar triangles) as CD to AB.

Cor. 2. *By reaſon of the great friction of the wedge, the power at* B, *muſt be to the reſiſtance againſt one ſide* AC; *at leaſt as twice the baſe* CD, *to the ſide* AC, *taking the reſiſtance perp. to* AC. *Or as twice the baſe* CD, *to the height* AB, *for the reſiſtance parallel to the baſe* CD; *to overcome the reſiſtance. But the power muſt be doubled for the reſiſtance againſt both ſides.*

For ſince the wedge retains any poſition it is driven into; therefore the friction muſt be at leaſt equal to the power that drives it.

Fig. 48. Cor. 3. *If you reckon the reſiſtance at both ſides of the wedge; then, if there is an equilibrium, the power at* B, *is to the whole reſiſtance; as the back* CD, *to the ſum of the ſides,* CA, AD, *reckoning the reſiſtance perp. to the ſides. Or as the back* CD, *to twice the height* AB, *for the reſiſtance parallel to the back* CD.

This follows directly from the Prop. and Cor. 1.

Scholium.

The principal uſe of the wedge is for the cleaving of wood, or ſeparating the parts of hard bodies, by the blow of a mallet. The force impreſſed by a mallet is vaſtly great in compariſon of a dead weight. For if a wedge, which is to cleave a piece of wood, be preſſed down with never ſo great a weight, or even if the other mechanical powers be applied to force it in; yet the effect of them will ſcarce be ſenſible; and yet the ſtroke of a ſledge or mallet will force it in. This effect is owing very much to the quantity of motion the mallet is put into, which it communicates in an inſtant to the wedge, by the force of percuſſion. A great deal of the reſiſtance is owing to friction, which hinders the motion of the wedge; but the ſtroke of a mallet overcomes it; upon which account the force of percuſſion is of excellent uſe; for a ſmart ſtroke puts the body into a tremulous vibrating motion, by which the parts are diſunited and ſeparated; and by this means the friction or ſticking is overcome, and the motion of the wedge made eaſy.

This mechanic power is the ſimpleſt of any; and to this, may be reduced all edge tools, as knives, axes, chiſſels, ſciſſars, ſwords, files, ſaws, ſpades, ſhovels, &c. which are ſo many wedges faſtened to a handle. And alſo all tools or inſtruments

ments with a ſharp point, as nails, bodkins, nee- Fig.
dles, pins; and all inſtruments to cleave, cut, ſlit, 48.
chop, pierce, bore, and the like. And in general all inſtruments that have an edge or point.

This Prop. is the ſame as Prop. XXX. in my large book of Mechanicks, but demonſtrated after a different way; and both come to the ſame thing, which evinces the truth thereof.

In this Prop. I have ſhewn under what circumſtance the wedge is in equilibrio; and that is, when the power is to the force againſt either ſide; as the back, is to that ſide. Therefore it muſt be very ſtrange, that any body ſhould underſtand it, as if I had ſaid, that the power is to the whole reſiſtance; as the back is to one ſide only. They that do this, muſt be blind or very careleſs.

Fig.

SECT. V.

The comparative Strength of Beams of Timber, and the Streſs they ſuſtain. The Powers of Engines, their Motion, and Friction.

PROP. LIV.

49. *IF a beam of wood* AB, *whoſe ſection is a parallellogram, be ſupported at the ends* A *and* B, *by two props* C, D. *And the weight* E *be laid on the middle of it, to break it; the ſtrength of it will be as the ſquare of the depth* EF, *when the breadth is given.*

For divide the depth EF into an infinite number of equal parts at *n*, *o*, *p*, *q*, *r*, &c. Now the ſtrength of the beam conſiſts of the ſtrength of all the fibres F*n*, *no*, *op*, &c. And to break theſe fibres, is to break the beam. Alſo when the beam is ſtretched by the weight, the fibres F*n*, *no*, *op*, &c. are ſtretched by the power of the bended levers AEF, AE*n*, AE*o*, &c. whoſe ſupport is at E, and power at A. For the preſſure at A being half the weight E, we muſt ſuppoſe that preſſure applied to A, to overcome the reſiſtances at F, *n*, *o*, &c. Put the force or preſſure at A = P, then P acts againſt all the fibres at F, *n*, *o*, &c. by help of the bended levers AEF, AE*n*, AE*o*, &c. But it is a known property of ſprings, fibres, and ſuch like expanding bodies; that the further they are ſtretched, the greater force they exert, in proportion to the length. Therefore when the beam breaks; that is, when the tenſion of the fibre F*n*

is

is at its utmoſt extent; then thoſe in the middle between F and E will have but half the tenſion, and thoſe at all other diſtances, will have a tenſion proportional to that diſtance. This being ſettled, let the utmoſt tenſion of nF be $= 1$; then the tenſions at n, o, p, &c. will be $\frac{En}{EF}$, $\frac{Eo}{EF}$, $\frac{Ep}{EF}$, &c. and the ſeveral forces, theſe exert againſt the point A, by means of the bended levers FEA, nEA, oEA, &c. will be $\frac{FE}{EA}$, $\frac{En^2}{EF \times EA}$, $\frac{Eo^2}{EF \times EA}$, $\frac{Ep^2}{EF \times EA}$ &c. and the ſum of all is $= \frac{1}{EF \times EA} \times$ into $EF^2 + En^2 + Eo^2 + Ep^2$ &c. to o. But (Arith. Inf. Prop. III.) the ſum of the progreſſion $EF^2 + En^2 + Eo^2$ &c. to o, is $\frac{1}{3}EF^3$. Therefore the ſum of all the forces exerted at A; that is, $P = \frac{1}{EF \times EA} \times \frac{1}{3}EF^3 = \frac{EF^2}{3EA}$. But $P = \frac{1}{2}$ weight E, therefore weight $E = \frac{2}{3} \times \frac{EF^2}{EA}$, when the beam breaks.

Fig. 49.

In like manner for any other depth Ep, the weight e that would break it is $= \frac{2Ep^2}{3EA}$. Whence the weight E to the weight e, is as EF^2 to Ep^2; that is, as the ſquares of the depths; for $\frac{2}{3EA}$ is a given quantity. Therefore the ſtrength of the beams, are as the ſquares of the depths.

Cor. 1. *Hence the ſtrength of ſeveral pieces of the ſame timber, are to one another as the breadths and ſquares of the depths.*

For by this Prop. they are as the ſquares of the depths when the breadth is given. And if the breadth be increaſed in any proportion, it is evi-

dent

Fig. 49. dent the ſtrength is encreaſed in the ſame proportion. So that a beam of the ſame depth being twice as broad is twice as ſtrong, and thrice as broad is thrice as ſtrong, &c.

50. Cor. 2. *If ſeveral beams of timber as* AF *of the ſame length, ſtick out of a wall; their ſtrength to bear any weight* W *ſuſpended at the end, is as the breadth and ſquare of the depth.*

This follows from Cor. 1. only turning the beam upſide down, to make the weight W ſuſpended at A act downwards inſtead of preſſing upwards.

49. Cor. 3. *If ſeveral pieces of timber be laid under one another, they will be no ſtronger than if they were laid ſide by ſide.*

For not being connected together in one ſolid piece, they can only exert each its own ſtrength, which will be the ſame in any poſition.

Cor. 4. *Hence the ſame piece of timber is ſtronger when laid edgeways or with the flat ſide up and down, than when laid flat ways, or with the flat ſide horizontal; and that in proportion of the greater breadth to the leſſer.*

For let B be the greater breadth, or the breadth of the flat ſide; *b* the leſſer breadth, being the narrow ſide. Then the ſtrength edge ways is BB*b*, and flat ways B*bb*; and they are to one another as B to *b*.

PROP. LV.

49. *If a beam of timber* AB *be ſupported at both ends; and a given weight* E *laid on the middle of it; the ſtreſs it ſuffers by the weight, will be as its length* AB.

For half the weight E is ſupported at A, by the prop C; and the preſſure at C is equal to it. And this preſſure is always the ſame whatever length

AB

AB is of. But it was ſhewn in the laſt Prop. that the preſſure at A, breaks the fibres F*n*, *no*, *op*, &c. by means of the bended levers AEF, AE*n*, AE*o*, &c. But (by Prop. XLV.) when the lengths EF, E*n*, E*o*, &c. are given, and the power at A alſo given; the effect at F, *n*, *o*, &c. is ſo much greater, as the arm AE is longer; that is, the ſtreſs at the ſection EF, is proportional to the diſtance AE, or to the length of the beam AB. Fig. 49.

Cor. 1. *If* AF *be a beam ſticking out of a wall, and a weight* W *hung at the end of it. The ſtreſs it ſuffers by the weight, at any point* G, *will be as the diſtance* AG. 50.

For this has the ſame effect, as in the caſe of this Prop. only turning the beam upſide down. Or thus, ſuppoſe AHG to be a bended lever, whoſe fulcrum is H; then ſince GH is given, and the weight W; therefore by the power of the lever, the longer AH is, the more force is applied at G, or any other points, in GH, to ſeparate the parts of the wood; and therefore the ſtreſs is as AG.

Cor. 2. *Therefore inſtead of a weight, if any force be applied at the end* A, *of the lever* AF; *the ſtreſs at any part* G, *will be as the force, and diſtance* AG.

For augmenting the force, the ſtreſs is increaſed in the ſame ratio.

Cor. 3. *Hence alſo if any weight lie upon the middle of a horizontal beam; the ſtreſs there will be as the weight and length of the beam.* 49.

For if the weight be increaſed, the ſtreſs will be increaſed proportionally, all other circumſtances remaining the ſame.

Cor. 4. *The ſtreſs of beams by their own weight, will be as the ſquares of the lengths.*

For here the weight is as the length.

PROP.

Fig.

PROP. LVI.

51. *If* AB *be a beam of timber whoſe length is given; and ſupported at the ends* A *and* B; *and if a given weight* W *be placed at any point of it* G. *The ſtreſs of the beam at* G, *will be as the rectangle* AGB.

Let the given weight be W. Then (Cor. 3. Prop. XLV.) the weight W is equal to the preſſure at both A and B. And (Cor. 2. Prop. XXXI.) preſſure at A : preſſure at B : : BG : AG, and preſſ. A : preſſ. A + preſſ. B : : BG : BG + AG; that is, preſſ. A : weight W : : BG : AB, therefore preſſure at $A = \frac{BG}{AB}W$, and this is the force re-acting at A. But (Prop. LV. Cor. 2.) the ſtreſs at G by this force acting at the diſtance AG, is as the force multiplied by AG; that is, as $\frac{AG \times BG}{AB} \times W$. But W and AB are given, and therefore the ſtreſs at G is as AG × GB.

Cor. 1. *The greateſt ſtreſs of a beam is when the weight lies in the middle.*

For the greateſt rectangle of the parts, is in that point.

Cor. 2. *The ſtreſs at any point* P *by a weight at* G; *is equal to the ſtreſs at* G, *by the ſame weight at* P.

For when the weight is at W, the ſtreſs at G is AG × GB, and the ſtreſs at $P = \frac{BP}{BG} \times$ the ſtreſs at $G = \frac{BP}{BG} \times AG \times GB = BP \times AG$. Again, when the weight is at P, the ſtreſs at P is AP × PB; and the ſtreſs at $G = \frac{AG}{AP} \times$ laſt ſtreſs $= \frac{AG}{AP} \times AP \times PB = AG \times PB$, the ſame as before.

PROP.

Fig.

PROP. LVII.

If the diſtance of the walls AD *and* BC *be given,* 52. *and* AB, AC *be two beams of timber of equal thickneſs; the one horizontal, the other inclined. And if two equal weights* P, Q, *be ſuſpended in the middle of them; the ſtreſs is equal in both, and the one will as ſoon break as the other, by theſe equal weights.*

For (Prop .XIV.) AC : AB : : weight P : $\frac{AB}{AC}P$ = preſſure againſt the plane, or the part of the weight the beam AC ſuſtains. And (Cor. 3. Prop. LV.) the ſtreſs upon AC is $\frac{AB}{AC}P \times AC$ or $AB \times P$; and the ſtreſs on AB is $Q \times AB$, which is equal to $AB \times P$, becauſe the weights P, Q are equal. Therefore, the ſtreſs being the ſame, and the beams being of equal thickneſs, one will bear as much as the other, and they will both break together.

Cor. 1. *If the beams be loaded with weights in any other places in the ſame perpendicular line as* F, G; *they will bear equal ſtreſs, and one will as ſoon break as the other.*

For they are cut into parts ſimilar to one another; and therefore ſtreſs at F : ſtreſs by P : : AFC : $\frac{1}{4}AC^2$: : AGB : $\frac{AB^2}{4}$: : ſtreſs by B : ſtreſs by Q or ſtreſs by P. Therefore ſtreſs at F = ſtreſs at B.

Cor. 2. *If the two beams be loaded in proportion to their lengths; the ſtreſs by theſe weights, or by their own weights, will be as their lengths; and therefore the longer, that ſtands aſlope, will ſooner break.*

For

Fig. 52. For the ſtreſs upon AC was AB × P, and the ſtreſs on AB was AB × Q; but ſince P and Q are to one another as AC and AB, therefore the ſtreſs on AC and AB will be as AB × AC and AB × AB; that is, as AC to AB. And in regard to their own weights, theſe are alſo proportional to their lengths.

PROP. LVIII.

53. *Let* AB, AC, *be two beams of timber of equal length and thickneſs, the one horizontal, the other ſet ſloping. And if* CD *be perp. to* AB, *and they be loaded in the middle with two weights* P, Q, *which are to one another as* AC *to* AD. *Then the ſtreſs will be equal in both, and one will as ſoon break as the other.*

For (Prop. XIV.) AC : AD : : P : $\frac{AD}{AC}P$ = preſſure of P in the middle of AC. And by ſuppoſition, AC : AD : : P : Q; therefore $\frac{AD}{AC}P$ = Q, the weight in the middle of AB. Therefore the forces in the middle of the two beams are the ſame; and the lengths of the beams being the ſame, therefore (Prop. LV.) the ſtreſs is equal upon both of them; and being of equal thickneſs, if one breaks, the other will break.

Cor. *If the weights* P, Q, *be equal upon the two equal beams* AB, AC. *The ſtreſs upon* AB *will be to the ſtreſs upon* AC, *as* AB *or* AC *to* AD. *The ſame holds in regard to their own weights.*

For the weight Q is increaſed in that proportion.

Scholium.

Many more propoſitions relating to the ſtrength of timber might be inſerted; as for example, if a weight

weight was disposed equally through the length of the beam AB (fig. 51.), supposed at both ends; the stress in any point G, is as the rectangle AGB. And the stress at any point G is but half of the stress it would suffer, if the whole weight was suspended at G. Also if AF (fig. 50.) be a beam fixed in a wall at one end, and a weight be dispersed uniformly through all the length of it. The stress at any point G, with that weight (or with its own weight, if it be all of a thickness), will be as AG square, the square of the distance from the end. And the stress at any point G by a weight suspended at A, will be double the stress at the same point G, when the same weight is dispersed uniformly through the part AG. They that would see these and such like things demonstrated, may consult my large book of Mechanics, to which I refer the reader. Fig. 53.

PROP. LIX.

If several pieces of timber be applied to any mechanical use where strength is required; not only the parts of the same piece, but the several pieces in regard to one another, ought to be so adjusted for bigness, that the strength may be always proportional to the stress they are to endure.

This Prop. is the foundation of all good Mechanism, and ought to be regarded in all sorts of tools and instruments we work with, as well as in the several parts of any engine. For who that is wise, will overload himself with his work tools, or make them bigger and heavier than the work requires? neither ought they to be so slender as not to be able to perform their office. In all engines, it must be considered what weight every beam is to carry, and proportion the strength accordingly. All levers

Fig. vers muſt be made ſtrongeſt at the place where they are ſtrained the moſt; in levers of the firſt kind, they muſt be ſtrongeſt at the ſupport. In thoſe of the ſecond kind, at the weight. In thoſe of the third kind, at the power, and diminiſh proportionally from that point. The axles of wheels and pullies, the teeth of wheels, which bear greater weights, or act with greater force, muſt be made ſtronger. And thoſe lighter, that have light work to do. Ropes muſt be ſo much ſtronger or weaker, as they have more or leſs tenſion. And in general, all the parts of a machine muſt have ſuch a degree of ſtrength as to be able to perform its office, and no more. For an exceſs of ſtrength in any part does no good, but adds unneceſſary weight to the machine, which clogs and retards its motion, and makes it languid and dead. And on the other hand, a defect of ſtrength where it is wanted, will be a means to make the engine fail in that part, and go to ruin. So neceſſary it is to adjuſt the ſtrength to the ſtreſs, that a good mechanic will never neglect it; but will contrive all the parts in due proportion, by which means they will laſt all alike, and the whole machine will be diſpoſed to fail all at once. And this will ever diſtinguiſh a good mechanic from a bad one, who either makes ſome parts ſo defective, imperfect and feeble as to fail very ſoon; or makes others, ſo ſtrong or clumſey, as to out laſt all the reſt.

From this general rule follows

Cor. 1. *In ſeveral pieces of timber of the ſame ſort, or in different parts of the ſame piece; the breadth multiplied by the ſquare of the depth, muſt be as the length multiplied by the weight to be borne.*

For then the ſtrength will be as the ſtreſs.

Cor. 2. *The breadth multiplied by the ſquare of the depth, and divided by the product of the length and weight, muſt be the ſame in all.*

Cor.

Cor. 3. *Hence may be computed the ſtrength of timber, proper for ſeveral uſes in building. As,* Fig.

1. To find the dimenſions of joiſts and boards for flooring. Let b, d, l be the breadth, depth and length of a joiſt, n = number of them, x = their diſtance, g = depth of a board, w = weight; then $nbdd$ = ſtrength of all the joiſts, and wl = ſtreſs on them, alſo $nlgg$ = ſtrength of the boards, and wx their ſtreſs; therefore $\frac{nbdd}{wl} = \frac{nlgg}{wx}$; and $x = \frac{llgg}{bdd}$, for the diſtance of the joiſts, or the length of a board between them. Or $b = \frac{llgg}{ddx}$, or $dd = \frac{llgg}{bx}$, and ſo on, according to what is wanted.

2. To find the dimenſions of ſquare timber for the roof of a houſe. Let r, s, l be the length of the ribs, ſpars and lats, ſo far as they bear; x, y, z their breadth or depth, n the diſtances of the lats, w = weight upon a rib, c = coſine of elevation of the roof. Then by reaſon of the inclined plane, $\frac{lw}{r} \times c$ = weight upon a ſpar. And $\frac{lnw}{rs}$ = weight upon a lat: for the ribs and lats lie horizontally. Therefore (Cor. 2.) $\frac{x^3}{wr} = \frac{y^3}{\frac{slw}{r} \times c} = \frac{z^3}{l \times \frac{lnw}{rs}}$.

Whence $x^3 = \frac{rry^3}{cls}$, and $x^3 = \frac{rrsz^3}{lln}$. Hence if any one x, y, or z be given, and all the reſt of the quantities; the other two may be found. Or in general, any two being unknown, they may be found, from having the reſt given.

For example, let $r = 9$ feet, $s = 4$ feet, $l = 15$ inches, $n = 11$ inches, $c = .707$ the coſine of 45°, the pitch of the roof. And aſſume $y = 2\frac{1}{2}$

Fig. inches; then $x = 2\frac{1}{2}\sqrt[3]{\frac{81}{3 \cdot 535}} = 7.1$ inches. And $z = y\sqrt[3]{\frac{ln}{css}} = 2\frac{1}{2}\sqrt[3]{\frac{55}{543}} = 1\frac{1}{6}$ inches.

54. 3. To find the curve ACB, into the form of which, if a joiſt be cut, on the upper or under ſide; and having the two ſides parallel planes, which are perp. to the horizon. That the ſaid joiſt ſhall be equally ſtrong every where to bear a given weight, ſuſpended on it.

Let the weight be placed in the ordinate CD; and the breadth of the beam, and the weight being given; then (Prop. LIV.) the ſtrength at C is as CD^2. And (Prop. LVI.) the ſtreſs is as ADB. Therefore that the ſtrength may be as the ſtreſs, CD^2 is as the rectangle ADB; and therefore the curve ACB is an ellipſis.

55. 4. To find the figure of a beam AB, fixed with one end in a wall, and having a given weight W ſuſpended at the other end B; and being every where of the ſame depth; it may be equally ſtrong throughout.

Let CD be the breadth at C; then (Prop. LIV.) the ſtrength is as CD. And (Cor. 1. Prop. LV.) the ſtreſs is as CB. Therefore CD is every where as CB, and therefore CDB is a plane triangle. And the beam is a priſm, whoſe upper and under ſides are parallel to the horizon.

56. 5. To find the figure of a beam AB, ſticking with one end in a wall, and of a given breadth; having a weight W ſuſpended at the end B; ſo that it may be equally ſtrong throughout.

Let CD be the depth at C. Then ſince the breadth is given, the ſtrength is as CD^2. And the ſtreſs as DB; therefore CD^2 is as DB. Whence CD is a common parabola.

57. 6. To find the figure of a beam AB, of the ſame breadth and depth, ſticking in a wall with one

one end, and bearing a weight ſuſpended at the other end B; ſo that it may be equally ſtrong throughout. Fig. 57.

Let CD be the thickneſs at O. Then the ſtrength is as CD^3, and the ſtreſs is as BO. Therefore BO is as CD^3 or as CO^3. And conſequently ACB is a cubic parabola, whoſe vertex is at B.

7. In like manner, if CDB be a beam fixed with one end in a wall, and all the ſides of it be cut into the form of a concave parabola, whoſe vertex is at B. It will be equally ſtrong throughout for ſupporting its own weight. 58.

For putting $BO = x$, $CO = y$, then by nature of the curve, $ay = xx$. But the ſolidity of CBD is $\frac{3.1416yyx}{5}$. And the center of gravity I, is diſtant $\frac{5}{6}x$ from B, therefore $OI = \frac{1}{6}x$. Now CD^3 or $8y^3$ = ſtrength at O. And CBD × OI or $\frac{3.1416yyx}{5} \times \frac{1}{6}x$ = ſtreſs. Therefore the ſtrength : to the ſtreſs : : is as $8y^3$: to $\frac{3.1416y^2xx}{30}$: : $8y$: $\frac{3.1416xx}{30}$: : $8y$: $\frac{3.1416ay}{30}$: : 340 : 3.1416a, that is, in a given ratio. And as as this happens every where, the ſolid is equally ſtrong in all parts.

I muſt take notice here that the 116th figure in my large book of Mechanics, is drawn wrong. It ſhould be concave inſtead of being convex.

8. Again, if AB be the ſpire of a church which is a ſolid cone or pyramid; it will be equally ſtrong throughout for reſiſting the wind. For the quantity of wind falling on any part of it ACD, will be as the ſection ACD. Therefore let $AO = x$, $CD = y$, and $x = ay$; then the ſtrength at $O = y^3$, and if I be the center of gravity of ACD, then 59.

Fig. 59. OI = $\frac{1}{3}x$. And the ſtreſs at O = wind ACD × OI = $xy \times \frac{1}{3}x$. Therefore the ſtrength is to the ſtreſs : : as y^3 : $\frac{1}{3}xzy$: : yy : : $\frac{1}{3}xx$ or $\frac{1}{3}aayy$: : 3 : aa; that is, in a given ratio. Therefore the ſpire is equally ſtrong every where.

Scholium.

It is all along ſuppoſed that the timber, &c. is of equal goodneſs, where theſe proportions for ſtrength are made. But if it is otherwiſe, a proper allowance muſt be made for the defect.

In theſe Propoſitions, I have called every thing *Strength*, that contributes in a direct proportion to reſiſt any force acting againſt a beam to break it; and I call *Streſs*, whatever weakens it in a direct proportion. But the whole may be referred to the article of ſtrength; for what I have called ſtreſs may be reckoned ſtrength in an inverſe ratio. Thus the ſtrength of a piece of timber may be ſaid to be directly as the breadth and ſquare of the depth, and inverſely as its length, and the weight or force applied; and that is equivalent to taking in the ſtreſs. But I had rather keep them diſtinct, and refer to each of them their proper effects, as I have all along done in the foregoing examples.

A piece of wood a foot long, and an inch ſquare, will bear as follows; oak from 320 to 1100; elm from 310 to 930; fir from 280 to 770 pounds, according to the goodneſs.

Fig.

PROP. LX.

In any machine contrived to raiſe great weights; if the power applied, be to the weight to be raiſed; as the velocity of the weight, to the velocity of the power; the power will only be in equilibrio with the weight. Therefore to raiſe it, the power muſt be ſo far increaſed, as to overcome all the friction and reſiſtance ariſing from the engine or otherwiſe; and then the power will be able to raiſe the weight.

A man would be much miſtaken, who ſhall make an engine to raiſe a great weight, and give his power no greater velocity, in regard to the velocity of the weight, than the quantity of the weight has in regard to the quantity of the power. For when he has done that, his weight and power will but have equal quantities of motion, and therefore they cannot ſet one another a moving, but muſt always remain at reſt. It is neceſſary then, that he do one of theſe two things. 1. That he apply a power greater than in that proportion, ſo much as to overcome all the friction and other accidental reſiſtance that may happen: and in ſome engines theſe are very great. Or 2. He muſt ſo continue his engine, that the velocity of the power, which ſuppoſe he has given, may be ſo much greater than the velocity of the weight; as the quantity of the weight, friction, and reſiſtance and all together, is greater than the power. This being done, the greater power will always overcome the leſſer, and his engine will work.

If a man does not attend to this rule, he will be guilty of many abſurd miſtakes, either in attempting things that are impoſſible, or in not applying means proper for the purpoſe. Hence it is that engines contrived for mines and water-works ſo often

Fig. 59. OI = $\frac{1}{3}x$. And the ſtreſs at O = wind ACD × OI = $xy \times \frac{1}{3}x$. Therefore the ſtrength is to the ſtreſs : : as y^3 : $\frac{1}{3}xzy$: : yy : : $\frac{1}{3}xx$ or $\frac{1}{3}aayy$: : 3 : aa; that is, in a given ratio. Therefore the ſpire is equally ſtrong every where.

Scholium.

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If a man does not attend to this rule, he will be guilty of many abſurd miſtakes, either in attempting things that are impoſſible, or in not applying means proper for the purpoſe. Hence it is that engines contrived for mines and water-works ſo often

Fig. ten fail ; as they muſt when either the quantity or velocity of the power is too little ; or which is the ſame thing, when the velocity of the weight is too great, and therefore would require more power than what is propoſed. As the weight is to move ſlow, the conſequence is, that it will be ſo much a longer time in moving through any ſpace. But there is no help for that. For as much as the weight to be raiſed is the greater, the time of raiſing it will be ſo much greater too.

Cor. 1. *Hence in raiſing any weight, what is gained in power is loſt in time. Or the time of riſing through any height will be ſo much longer as the weight is greater.*

If the power be to the weight as 1 to 20; then the ſpace through which the weight moves will be 20 times leſs, and the time will be 20 times longer in moving through any ſpace, than that of the power. The advantage that is gained by the ſtrength of the motion, is loſt in the ſlowneſs of it. So that though they increaſe the power, they prolong the time. And that which one man may do in 20 days, may be done by the ſtrength of twenty men in one day.

Cor. 2. *The quantity of motion in the weight is not at all increaſed by the engine. And if any given quantity of power be immediately applied to a body at liberty, it will produce as much motion in it, as it would do by help of a machine.*

PROP. LXI.

If an engine be compoſed of ſeveral of the ſimple mechanic powers combined together ; it will produce the ſame effect, ſetting aſide friction ; as any one ſimple mechanic power would do, which has the ſame power or force of acting.

For let any compound engine be divided into all the ſimple powers that compoſe it. Then the force

or

or power applied to the firſt part, will cauſe it to act upon the ſecond with a new power, which would be deemed the weight, if the machine had no more parts. This new power acting on the ſecond part, will cauſe it to act upon the third part: and that upon a fourth, and ſo on till you come at the weight, which will be acted on, by all theſe mediums, juſt the ſame as by a ſimple machine, whoſe power is equal to them all. Fig.

Cor. 1. *Hence a compound machine may be made, which ſhall have the ſame power, as any ſingle one propoſed.*

For if a lever is propoſed whoſe power is 100 to 1; two levers acting on one another will be equivalent to it, where the power of the firſt is as 10 to 1, and that of the ſecond alſo as 10 to 1, or the firſt 20 to 1, and the ſecond 5 to 1; or any two numbers, whoſe product is 100.

Again, a wheel and axle whoſe power is as 48 to 1, may be reſolved into two or more wheels with teeth, to have the ſame power; for example, make two wheels, ſo that the firſt wheel and pinion be as 8 to 1, and the ſecond as 6 to 1. They will have the ſame effect as the ſingle one. Or break it into three wheels, whoſe ſeveral powers may be 4 to 1, and 4 to 1, and 3 to 1.

If a ſimple combination of pullies be as 36 to 1; you may take three combinations to act upon one another, whoſe powers are 3 to 1, 3 to 1, and 4 to 1.

And after the ſame manner it is to be done in machines more compounded.

And this is generally done to ſave room. For when an engine is to have great power, it is hardly made of one wheel, it would be ſo large; but by breaking it into ſeveral wheels, after this manner, it will go into a little room, and have the

Fig. ſame power as the other. All the inconvenience is, it will have more friction; for the more parts acting upon one another, the more friction is made.

Cor. 2. *Hence alſo it follows, that in any compound machine, its power is to the weight, in the compound ratio of the power to the weight in all the ſimple machines that compoſe it.*

Cor. 3. *Hence it will be no difficult matter to contrive an engine that ſhall overcome any force or reſiſtance aſſigned.*

For if you have the quantity of power given, as well as of the weight or reſiſtance; it is but taking any ſimple machine, as a lever, wheel, &c. ſo that the power may be to the weight in the ratio aſſigned, adding as much to the weight as you judge the friction will amount to. When this ſimple machine is obtained; break it or reſolve it into as many other ſimple ones as you think proper; ſo that they may have the ſame power.

And as to the ſeveral ſimple machines, it matters not what ſort they are of, as to the power; whether they be levers, wheels, pullies, or ſcrews; but ſome are more commodious than others for particular purpoſes; which a mechanic will find out beſt by practice. In general, a lever is the moſt ready and ſimple machine to raiſe a weight a ſmall diſtance; and for further diſtances, the wheel and axle, or a combination of pullies; or the perpetual ſcrew. Alſo theſe may be combined with one another; as a lever with a wheel or a ſcrew, the wheel and axle with pullies, pullies with pullies, and wheels with wheels, the perpetual ſcrew and the wheel. But in general a machine ſhould conſiſt of as few parts as is conſiſtent with the purpoſe it is deſigned for, upon account of leſſening the friction; and to make it ſtill leſs, the joints muſt

be

be oiled or greaſed. All parts that act on one another muſt be poliſhed ſmooth. The axles or ſpindles of wheels muſt not ſhake in the holes, but run true and even. Likewiſe the larger a machine is, if it be well executed, the better and truer it will work. And large wheels and pullies, and ſmall axles or ſpindles have the leaſt friction. Fig.

The power applied to work the engine may be men or horſes; or it may be a weight or a ſpring; or wind, water, or fire; of which one muſt take that which is moſt convenient and coſts the leaſt. Wind and water are beſt applied to work large engines, and ſuch as muſt be continually kept going. A man may act for a while againſt a reſiſtance of 50 pounds; and for a whole day againſt 30 pounds. A horſe is about as ſtrong as five men.

If two men work at a roller, the handles ought to be at right angles to one another.

When a machine is to go regular and uniform, a heavy wheel or fly muſt be applied to it.

Scholium.

Two things are required to make a good engineer. 1. A good invention for the ſimple and eaſy contrivance of a machine, and this is to be attained by practiſe and experience. 2. So much theory as to be able to compute the effect any engine will have; and this is to be learned from the principles of Mechanics.

PROP. LXII.

The friction or reſiſtance ariſing by a body moving upon any ſurface, is as the roughneſs of the ſurface, and nearly as the weight of the body; but is not much increaſed by the quantity of the ſurface of the moving body, and is ſomething greater with a greater velocity.

It is matter of experience that bodies meet with a great deal of reſiſtance by ſliding upon one another,

Fig. ther, which cannot be entirely taken away, though the bodies be made never ſo ſmooth: yet by ſmoothing or poliſhing their ſurfaces, and taking off the roughneſs of them, this reſiſtance may be reduced to a ſmall matter. But many bodies, by their natural texture, are not capable of bearing a poliſh; and theſe will always have a conſiderable degree of reſiſtance or friction. And thoſe that can be poliſhed, will have ſome of this reſiſtance ariſing from the coheſion of their ſurfaces. But in general, the ſmoother or finer their ſurfaces, the leſs the friction will be.

As the ſurfaces of all bodies are in ſome degree rough and uneven, and ſubject to many inequalities; when one body is laid upon another, the prominent parts of one fall into the hollows of the other; ſo that the body cannot be moved forward, till the prominent parts of one be raiſed above the prominent parts of the other, which requires the more force to effect, as theſe parts are higher; that is, as the body is rougher. And this is ſimilar to drawing a body up an inclined plane, for theſe protuberances are nothing elſe but ſo many inclined planes, over which the body is to be drawn. And therefore the heavier the body, the more force is required to draw it over theſe eminences; whence the friction will be nearly as the weight of the body.

But whilſt the roughneſs remains the ſame, or the prominent parts remain of the ſame height, there will always be required the ſame force, to draw the ſame weight. And the increaſing of the ſurface, retaining the ſame weight, can add nothing to the reſiſtance on that account; but it will make ſome addition upon other accounts. For when one ſurface is dragged along another, ſome part of the reſiſtance ariſes from ſome parts of the moving ſurface, taking hold of the parts of the other, and tearing them off; and this is called *wearing*. And therefore

therefore this part of the friction is greater in a greater ſurface, in proportion to that ſurface. There is likewiſe in a greater ſurface, a greater force of coheſion, which ſtill adds ſomething to the friction. But the two parts of the friction, ariſing from the wearing and tenacity, are not increaſed by the velocity: but the other part, of drawing them over inclined planes, will increaſe with the velocity. So that in the whole, the friction is ſomewhat increaſed by the quantity of the ſurface, and by the velocity, but not much. But more in ſome bodies than others, according to their particular texture. Fig.

Cor. 1. *Hence there can be no certain rule, to eſtimate the friction of bodies; this is a matter that can only be decided by experiments. But it may be obſerved, that,* cæteris paribus, *hard bodies will have leſs reſiſtance than ſofter; and bodies oiled or greaſed, will have far leſs.*

For the particles of hard bodies, cannot ſo well take hold of one another to tear themſelves off. And when a ſurface is oiled, it is the ſame thing as if it run upon a great number of rollers or ſpheres.

Cor. 2. *Hence alſo a method appears of meaſuring the friction of a body ſliding upon another body, by help of an inclined plane.*

Take a plank CB of the ſame matter, raiſe it at one end C ſo high, till the body whoſe friction is ſought, being laid at C, ſhall juſt begin to move down the plane CB. Then the weight of the body, is to the friction, as the baſe AB to the height AC of the plane. For the preſſure againſt the plane is the part of the weight that cauſes the friction, and the tendency down the plane is equal to the friction. And (Prop. XIV.) that preſſure is to the tendency as AB to AC. 60.

If

Fig. 60. If you puſh the body from C downward, and obſerve it to keep the ſame velocity through D to B; then you will have the friction for that velocity. If it increaſes its velocity, lower the end of the plank C; if it grows ſlower, raiſe the end C till you get the body to have the ſame velocity quite through the plane. And ſo you will find what elevations are proper for each velocity; and from thence the ratio of AB to AC, or of the weight to the friction.

There is a way to make the experiment, by drawing the body along a horizontal plane, by weights hung at a ſtring, which goes over a pulley; but the method here deſcribed is more eaſy and ſimple.

SCHOLIUM.

From what has been before laid down, it will be eaſy to underſtand the nature of engines, and how to contrive one for any purpoſe aſſigned. And likewiſe having any engine before us, we can by the ſame rules, compute its powers and operations.

Engines are of various kinds; ſome are fixed in a particular place, where they are to act; as windmills and water-mills for corn, fire engines for drawing water, gins for coal pits, many ſorts of mills; pumps, cranes, &c. others are movable from one place to another, and may be carried to any place where they are wanted, as blocks, pullies and tackles for raiſing weights, the lifting jack, and lifting ſtock, clocks, watches, ſmall bellows, ſcales, ſteelyards, and an infinite number of others. Another ſort of engines are ſuch as are made on purpoſe to move from one place to another, ſuch as boats, ſhips, coaches, carriages, waggons, &c. If any of theſe are urged forward by the help of levers, wheels, &c. By having the acting power given, the moving force that drives it forward, is eaſily found by the properties of

theſe machines. Only obſerve, if the firſt acting power be external, as wind, water, horſes, &c. you muſt not forget to add or ſubtract it, to or from the moving force before found; according as that firſt acting power conſpires with, or oppoſes the motion of the machine; and the reſult is the true force it is driven forward with. I have only room to deſcribe a very few engines, but thoſe that deſire it may ſee great variety in my large book of Mechanics. Fig.

A WHEEL CARRIAGE.

AB is a cart or carriage, going upon two wheels as CD, and ſometimes upon four, as all waggons do. The advantages of wheel carriages is ſo great, that no body who has any great weight to carry, will make uſe of any other method. Was a great weight to be dragged along upon a ſledge or any ſuch machine without wheels, the friction would be ſo great, that a ſufficient force in many caſes could not be got to do it. But by applying wheels to carriages, the friction is almoſt all of it taken away. And this is occaſioned by the wheel's turning round upon the ground, inſtead of dragging upon it. And the reaſon of the wheel's turning round is the reſiſtance the earth makes againſt it at O where it touches. For as the carriage goes along, the wheel meets with a reſiſtance at the bottom O, where it touches the ground; and meeting with none at the top at C, to balance it; that force at O muſt make it turn round in the order ODC, ſo that all the parts of the circumference of the wheel are ſucceſſively applied to the earth. In going down a ſteep bank it is often neceſſary to tie one wheel faſt, that it cannot turn round, this will make it drag; and by the great reſiſtance it meets with, ſtops the too violent motion, the carriage would otherwiſe have, in deſcending the hill. 61.

But

Fig. 61. But although all ſorts of wheels very much diminiſh the friction; yet ſome have more then others; and it may be obſerved that great wheels, and ſmall axles have the leaſt friction. To make the friction as little as poſſible, ſome have applied friction wheels, which is thus; EG is the friction wheel running upon an axis I which is fixed in the piece of timber ES, which timber is fixed to the ſide of the carriage. KL is the axle of the carriage, which is fixed in the wheel CD, ſo that both turn round together. Then inſtead of the carriage lying upon the axle KL, the friction wheel FG lies upon the axle; ſo that when the wheel CD turns round, the axle cauſes the friction wheel, with the weight of the carriage upon it, to turn round the center I, which diminiſhes the friction in proportion to the radius IG: and there is the ſame contrivance for the wheel on the other ſide. But the wheel CD need not be fixed to the axle; for it may turn round on the axle KL, and alſo the axle turn round under the carriage.

In paſſing over any obſtacles, the large wheels have the advantage. For let MN be an obſtacle; then drawing the wheel over this obſtacle, is the ſame thing as drawing it up the inclined plane MP, which is a tangent to the point M; but the greater the wheel CD is, the leſs is that plane inclined to the horizon.

Likewiſe great wheels do not ſink ſo deep into the earth as ſmall ones, and conſequently require leſs force to pull them out again.

But there are diſadvantages in great wheels; for in the firſt place they are more eaſily overturned; and ſecondly, they are not ſo eaſy to turn with, in a ſtrait road as ſmall wheels.

The tackle of any carriage ought to be ſo fixed, that the horſe may pull partly upwards, or lift, as well as pull forwards; for all hills and inequalities.

in the road, being like ſo many inclined planes, Fig. the weight is moſt eaſily drawn over them, when 61. the power draws at an equal elevation.

A carriage with four wheels is more advantageous, than one with two only, but they are bad to turn; and therefore are obliged to make uſe of ſmall fore wheels. Broad wheels which are lately come into faſhion, are very advantageous, as they ſink but little into the earth. But there is a diſadvantage attends them, for they take up ſuch a quantity of dirt by their great breadth, as ſenſibly retards the carriage by its weight, and the like may be ſaid of their own weight.

The under ſide of the axle where the wheels are, muſt be in a right line; otherwiſe if they ſlant upwards, the weight of the carriage will cauſe them to work toward the end, and preſs againſt the runners and lin pin. And as the ends of the axle are conical, this cauſes the wheels to come nearer together at bottom, and be further diſtant at the top; by which means the carriage is ſooner overturned. To help this, the ends of the axle muſt be made as near a cylindrical form as poſſible, to get the wheels to fit, and to move free.

A HAND MILL.

Fig 62. is a hand mill for grinding corn, A, B 62. the ſtones included in a wooden caſe. A the upper ſtone, being the living or moving ſtone. B the lower ſtone, or the dead ſtone, being fixed immovable. The upper ſtone is 5 inches thick, and a foot and three quarters broad; the lower ſtone is broader. C is a cog-wheel, with 16 or 18 cogs; DE its axis. F is a trundle with 9 rounds, fixed to the axis G, which axis is fixed to the upper ſtone A, by a piece of iron made on purpoſe. H is the hopper, into which the corn is put; I the ſhoe, to carry the corn by little and little through a hole

at

Fig. 62. at K, to fall between the two ſtones. L is the mill eye, being the place where the flour or meal comes out after it is ground. The under ſtone is ſupported by ſtrong beams not drawn here. And the ſpindle G ſtands on the beam MN, which lies upon the bearer O, and O lies upon a fixed beam at one end, and at the other end has a ſtring fixed, and tied to the pin P. The under ſtone is not flat, but riſes a little in the middle, and the upper one is a little hollow. The ſtones very near touch at the outſide, but are wider towards the middle to let the corn go in.

When corn is to be ground, it is put into the hopper H, a little at a time, and a man turns the handle D, which carries round the cog-wheel C, and this carries about the trundle F, and axis G, and ſtone A. The axis G is angular at K; and as it goes round, it ſhakes the ſhoe I, and makes the corn fall gradually through the hole K. And the upper ſtone going round grinds it, and when ground it comes out at the mill eye L, where there is a ſack or tub placed to receive it. Another handle may be made at E like that at D, for two men to work, if any one pleaſes. In order to make the mill grind coarſer or finer, the upper ſtone A may be lowered or raiſed, by means of the ſtring going from the bearer O; for turning round the pin P, the ſtring is lengthened or ſhortened, and thereby the timbers O, M are lowered or raiſed, and with them the axle G and ſtone A. For the ſpindle G goes through the ſtone B, and runs upon the beam MN. The ſpindle is made ſo cloſe and tight, by wood or leather, where it goes through the under ſtone, that no meal can fall through. The under ſide of the upper ſtone is cut into gutters in the manner repreſented at Q. It is a pity ſome ſuch like mills are not made at a cheap rate for the ſake of the poor, who are much diſtreſſed by the roguery of the millers.

Fig.

Fig. 63. is a ſort of crane, BC an upright poſt, AB a beam fixed horizontally at top of it; theſe turn round together on the pivot C, and within the circle S, which is fixed to the top of the frame PQ. EF is a wooden roller, or rather a roller made of thin boards, for lightneſs, and all nailed to ſeveral circular pieces on the inſide. GH a wheel fixed to the roller, about which goes the rope GR. IK, LN, two other ropes; fixed with one end to the croſs piece AB, and the other end to the roller EF. W a weight equal to the weight of the wheel and roller, which is faſtened to a rope which goes over the pulley O, and then is faſtened to a collar V, which goes round the roller. ET is another rope with a hook at it to lift up any weight, the other end of the rope being fixed to the roller; here are in all five ropes. Fig. 63.

To raiſe any weight as M, hang it upon the hook T, then pulling at the rope R which goes about the wheel GH, this cauſes the wheel and roller to turn round, and the ropes IK, LN to wind about it, by which means the wheel and axle riſes; and by riſing, folds the rope TE about the roller the contrary way, and ſo raiſes the weight M. When the weight M is raiſed high enough, a man muſt take hold of the rope T with a hook, by which the whole machine may be drawn about, turning upon the centers C and S. And then the weight M may be let down again. The weight of the wheel and roller do not affect the power drawing at R, becauſe it is balanced by the weight W. There is no friction in this machine but what is occaſioned by the collar V, and the bending of the ropes. And the power is to the weight in this crane, as the diameter of the roller to the radius of the wheel GH.

Fig.

An ENGINE for raiſing WEIGHTS.

64. Fig. 64. is an engine compoſed of a perpetual ſcrew AB, and a wheel DE with teeth, and a ſingle pulley H. FG is an axle, about which a rope goes, which lifts the pulley and weight W. BC is the winch, to turn it round withal. As the ſpindle AB is turned about, the teeth of it takes the teeth of the wheel DE, and turns it about, together with the axle FG, which winds up the rope, and raiſes the pulley H, with the weight W. The power at C, is to the weight W, as diameter FG × by the breadth of one tooth, is to twice the diameter DE × circumference of the circle deſcribed by C.

A FULLING MILL.

65. Fig. 65. is a fulling mill. AB a great water wheel, carried about by a ſtream of water, coming from the trough C, and falling into the buckets D, D, D whoſe weight carries the wheel about; this is a breaſt mill, becauſe the water comes no higher than the middle or breaſt of the wheel; EF is its axis; I, I; K, K, two lifters going through the axle, which raiſe the ends G, G of the wooden mallets GH, GH, as the wheel goes about; and when the end G ſlips off the cog or lifter K or I, the mallet falls into the trough L, and each of the mallets makes two ſtrokes for one revolution of the wheel. The mallets move about the centers M, M. Theſe troughs L, L, contain the ſtuff which is to be milled, by the beating of the mallets. N, N, is a channel to carry the water, being juſt wide enough to let the wheel go round. And the wheel may be ſtopt, by turning the trough C aſide, which brings the water. In this engine more mallets may be uſed, and then more pins or lifters muſt be put through the axis EF.

Fig.

Fig. 66. is a common *Pocket Watch.* AA the *balance*, BB the *verge*; C, C, two *palats.* D the *crown wheel* acting againſt the palats C, C; E its pinion. F the *contrate wheel*, G its pinion. H the *third wheel*, I its pinion. K the ſecond wheel or *center wheel*, L its pinion. M the *great wheel*, N the *fuſee* turning round upon the ſpindle of M. O the *ſpring box*, having a ſpring included in it. PP the *chain* going round the ſpring box O, and the fuſee N. This work is within the watch between the two plates. Here the face is downward, and in the watch the wheel K is placed in the center, and the others round about it. Here I have placed them ſo as beſt to be ſeen, which ſignifies nothing to the motion. The balance AA is without the plate, covered by the cock X. The minute hand Q goes upon the axis of the wheel K. Fig. 66.

Then between the upper plate and the face, we have V the *cannon pinion* or *pinion of report.* Z the *dial wheel.* T the *minute wheel.* S the *pinion* or *nut*, fixed to it. The ſocket of the cannon pinion V goes into the ſocket of the wheel Z, and are movable about one another, and both go through the face; on the ſocket of the pinion Z is fixed the hour hand R; and on the ſocket of V is fixed the minute hand Q. Likewiſe the ſocket of V is hollow, and both go upon the arbor of the wheel K, which reaches through the face, and are faſtened there. The wheel and ſocket, T, S are hollow, and go upon a fixed axle on which they turn round.

When the chain PP is wound up, upon the fuſee N; the ſpring included in the box O, draws the chain PP, which forces about the wheel M, the fuſee being kept from ſlipping back, by a catch on purpoſe. Then M drives L and K, and K drives I, and H drives G, and F drives E, and the teeth of the crown wheel D, act againſt the palats C, C alternately, and cauſe the balance AA

Fig. to vibrate back and forward, and thus the watch is kept going.

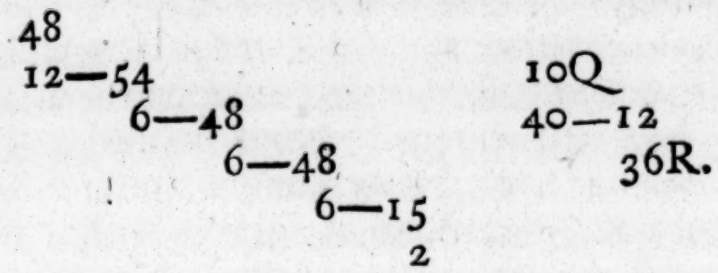
66. The cannon pinion and dial wheel V and Z, and the hands Q, R, being put upon the arbor of K at W; and faſtened there by means of a ſhoulder which is upon the axis, and a braſs ſpring; as the wheel K goes round, it carries with it the pinion V with the minute hand, and V drives T together with S; and S drives Z with the hour hand.

The numbers of the wheels and pinions, (that is the teeth in them) are, M = 48, L = 12, K = 54, I = 6, H = 48, G = 6, F = 48, E = 6, D = 15, and 2 palats. The *train*, or number of beats in an hour, is 17280, which is about $4\frac{1}{4}$ beats in a ſecond. Alſo V = 10, Z = 36, S = 12, T = 40.

The wheel M goes round 6 times in 24 hours, therefore K goes round $\left(\frac{48}{12}\right)$ 4 times as much; that is, 24 times, or once in an hour, and the hand Q along with it; therefore Q will ſhew minutes. Then as V goes round once in an hour, T will go round $\left(\frac{10}{40}\right)$ $\frac{1}{4}$ of that, or $\frac{1}{4}$ the circumference; and as S goes $\frac{1}{4}$, Z will go $\left(\frac{12}{36}\right)$ $\frac{1}{3}$ of that, or $\frac{1}{12}$ of the circumferenee in an hour, and therefore as R goes along with it, R will ſhew the hours. The wheels and pinions T, Z, and S, V, are drawn with the face upwards. And the whole machine included in a caſe is but about two inches diameter. There is a ſpiral ſpring fixed under the balance AB, called the *regulator*, which gives it a regular motion; and likewiſe abundance of ſmall parts helpful to her motion, too long to be deſcribed here.

The

The way of writing down the numbers, is thus, Fig. 66.



Explanation. The wheel with 48 drives a pinion of 12, and a wheel of 54 on the ſame arbor. The wheel 54 drives the pinion 6 with the wheel 48 on the ſame arbor. The wheel 48 drives the pinion 6 and wheel 48 on the ſame arbor. The wheel 48 drives the pinion 6 and wheel 15 on the ſame arbor. And the wheel 15 drives the two palats.

Again the wheel 54 has the pinion 10 on its arbor, and the hand Q; and the pinion 10 drives the wheel 40, with the pinion 12. And the pinion 12 drives the wheel 36 with the hand R.

As this machine is moved by a ſpring, it is ſubject to very great inequalities of motion, occaſioned by heat and cold. For hot weather ſo relaxes, ſoftens, and weakens the main ſpring, that it loſes a great deal of its ſtrength, which cauſes the watch to loſe time and go too ſlow. On the other hand, cold froſty weather ſo affects the ſpring, and it is ſo condenſed and hardened, that it becomes far ſtronger; and by that means accelerates the motion of the watch, and makes her go faſter. The difference of motion in a watch, thus occaſioned by heat and cold, will often amount to an hour, and more in 24 hours. To remedy this, there is a piece of machinery, called the *Slide*, placed near the regulating ſpring; which being put forward or backward, ſhortens or lengthens the ſpring, ſo as to make her keep time truly.

Some people have been ſo ſilly as to think, that the greater ſtrength of a ſpring ariſes wholly

Fig. from its being made ſhorter, as this happens to be one of the effects of cold. But it is eaſily demon-
67. ſtrated that this is not the cauſe. For let AB be a ſpring as it is dilated by heat, and *ab* the ſame ſpring contracted by cold. Now if the ſpring has been contracted in length, it muſt be proportionally contracted in all dimenſions. Let *l*, *b*, *d*, denote the length, breadth, and depth, in its cold, and leaſt dimenſions; and *rl*, *rb*, *rd*, the length, breadth, and depth, in its hot and greateſt dimenſions. Then (Prop. LIV.) the ſtrength of the longer, to the ſtrength of the ſhorter, will be as $\frac{rb \times rrdd}{rl}$ to $\frac{bdd}{l}$ (conſidering it weakened by the length) and that is as *rr* to 1, or as AB^2 to ab^2. So that the longer ſpring, upon account of its being affected with heat, is ſo far from being weaker, than the ſhorter affected with cold, that it is the ſtronger of the two. And therefore this difference is not to be aſcribed merely to the lengthning or ſhortning thereof; but muſt be owing to the nature, texture and conſtitution of the ſteel, as it is ſome way or other affected and changed by the heat and cold.

And that there is ſome change induced by the cold, into the very texture of the metal, it is evident from this, that all ſorts of tools made of iron or ſteel, as ſprings, knives, ſaws, nails, &c. very eaſily ſnap and break in cold froſty weather, which they will not do in hot weather. And that property of ſteel ſprings is the true cauſe, that theſe ſorts of movements can never go true.

66. To make a calculation of the different forces requiſite to make a watch gain or loſe any number of minutes, as ſuppoſe half an hour in 24; and I have often experienced it to be more. By Cor. 4. Prop. VI. the product of the force and ſquare of the

the time, is as the product of the body and ſpace deſcribed, which here is a given quantity. For the matter of the balance remains the ſame in hot as cold weather; and ſo does the length of the ſwing, which here is the ſpace deſcribed. Therefore the force is reciprocally as the ſquare of the time of vibrating, or directly as the ſquare of the number of vibrations in 24 hours. Therefore the force with the warm ſpring, is to the force with the cold one; as the ſquare of 23 ½ hours, to the ſquare of 24; that is, nearly as 23 to 24. So that if a ſpring was to contract half an inch in a foot in length, without altering its other dimenſions, it would but be ſufficient to account for that phænomenon; but this is forty times more than the lengthening and ſhortening by heat and cold, for that does not alter ſo much as a thouſandth part, as is plain from experiments. Fig. 66.

The caſe being thus, a clock or watch going by a ſpring, can never be made to keep time truly, except it be always kept to the ſame degree of heat or cold, which cannot be done without conſtant attendance. And if any ſort of mechaniſm be contrived to correct this; yet as ſuch a thing can only be made by gueſs, it cannot be truſted to at ſea, but only for ſhort voyages. But no motion however regular, can ever anſwer at ſea, where the irregular motion of the ſhip will continually diſturb it; add to this, that the ſmall compaſs a watch is contained in, makes it eaſier diſturbed, than a larger machine would be; but to ſuppoſe that any regular motion can ſubſiſt among ten thouſand irregular motions, and in ten thouſand different directions, is a moſt glaring abſurdity. And if any one with ſuch a machine would but make trial of it to the Eaſt Indies, he would find the abſurdity and diſappointment. And therefore I never expect to ſee ſuch a time keeper, or any ſuch thing as a watch or clock

Fig. 66. going by a ſpring, to keep true time at ſea. But time will diſcover all things.

As to pendulum clocks, their irregularity in the ſame latitude is owing to nothing but the lengthning or ſhortning of the pendulum; which is a mere trifle to the other. But then they would be infinitely more diſturbed at ſea, than a watch; and in a ſtorm could not go at all. In different latitudes too, another irregularity attends a pendulum, depending on the different forces of gravity. Though this amounts but to a ſmall matter, yet it makes a conſiderable variation, in a great length of time. For in ſouth latitudes, where the gravity is leſs, a clock loſes time. And in north latitudes, where the gravity is greater, it gains time. So that none of theſe machines are fit to meaſure time at ſea, although ten times ten thouſand pounds ſhould be given away for making them.

A DESCENDING CLOCK.

68. Fig. 68. is a clock deſcending down an inclined plane. This conſiſts of a train of watch work, contained between two circular plates AB, CD, 4 inches diameter, fixed together by a hoop an inch and half broad, incloſing all the work. The inner work conſiſts of 5 wheels, the ſame as in a watch, only there is a ſpur wheel inſtead of the contrate wheel, as 4; *b* is the balance, whoſe palats play in the teeth of the crown wheel 5. Here is no ſpring to give it motion, but inſtead thereof, the weight W is fixed to the wheel 1, and ſo adjuſted for weight, that it may balance the lower ſide, and hinder it from rolling down the plane. Now whilſt the weight W moves the wheel 1, this wheel by moving about, cauſes the weight W to deſcend, by which it ceaſes to be a balance for the oppoſite ſide, and therefore that ſide begins to deſcend,

ſcend, till the weight W be raiſed high enough again to become a balance, which muſt be about the poſition it appears in the figure. Thus whilſt wheels move gradually about, the weight W deſcends gradually, which makes the body of the machine turn gradually round, and deſcend down the inclined plane PQ; making one revolution in 12 hours. And therefore to have her to go 24 or 30 hours; the length of the plane PQ muſt be 2 or $2\frac{1}{2}$ circumferences of the plates. Before the weight W is fixed to the wheel 1, ſome lead or braſs muſt be ſoldered on the ſide E oppoſite to the wheels 2, 3, 4, &c. for the wheel 1 muſt be in the center. And then the lead or braſs muſt be filed away till the center of gravity of the machine be in the center of the plates. And to hinder the machine from ſliding, the edges of the plates muſt be lightly indented. The inclined plane PQ may be a board, which muſt be elevated 10 or 12 degrees, but that is to be found by trials; for if ſhe go too ſlow the end P muſt be raiſed; but if too faſt it muſt be lowered. When the clock has gone the length of the board to Q, it muſt be ſet again at P. The fore ſide CD is divided into hours, and a pin is fixed in the center at G, on which the hand FGH, always hangs looſely in a perp. poſition, with the heavy end H downward. And the end F ſhews the hour of the day. So that the hours come to the hand, and not the hand to the hours. Fig. 68.

The board PQ muſt be perfectly ſtraight from one end to the other, or elſe ſhe will go faſter in ſome places and ſlower in others.

The circle with hours ought to be a narrow rim of braſs, movable round about, by the help of one or more pins placed in it; ſo that it may be ſet to the true time.

The

Fig. 68. The weight W ſerves for two uſes, 1, to be a counterpoiſe to the ſide A; and 2, by its weight to put the clock in motion.

The weight W muſt be ſo heavy as to make the clock keep time, when it has a proper degree of elevation, as 45 degrees; and then the board muſt have an elevation of 10 or 12 degrees. If ſhe go too faſt, with theſe poſitions, take ſomething off the weight; if too ſlow, add ſomething to it.

SECT.

Fig.

SECT. VI.

HYDROSTATICS *and* PNEUMATICS.

DEFINITION I.

A *FLUID* is ſuch a body whoſe parts are eaſily moved among themſelves, and yield to any force acting againſt them.

DEF. II.

Hydroſtatics, is a ſcience that demonſtrates the properties of fluids.

DEF. III.

Hydraulics, is the art of raiſing water by engines.

DEF. IV.

Pneumatics, is that ſcience which ſhews the properties of the air.

DEF. V.

A fountain or *jet d'eau*, is an artificial ſpout of water.

PROP. LXIII.

If one part of a fluid be higher than another, the higher parts will continually deſcend to the lower places, and will not be at reſt, till the ſurface of it is quite level.

For the parts of a fluid being movable every way, if any part is above the reſt, it will deſcend by its own gravity as low as it can get. And afterwards other parts that are now become higher, will

Fig. will deſcend as the other did, till at laſt they will all be reduced to a level or horizontal plane.

Cor. 1. *Hence water that communicates by means of a channel or pipe, with other water, will ſettle at the ſame height in both places.*

Cor. 2. *For the ſame reaſon, if a fluid gravitates towards a center; it will diſpoſe itſelf into a ſpherical figure, whoſe center is the center of force. As the ſea in reſpect of the earth.*

PROP. LXIV.

If a fluid be at reſt in a veſſel whoſe baſe is parallel to the horizon; equal parts of the baſe are equally preſſed by the fluid.

For upon every part of the baſe there is an equal column of the fluid ſupported by it. And as all theſe columns are of equal weight, they muſt preſs the baſe equally; or equal parts of the baſe will ſuſtain an equal preſſure.

Cor. 1. *All parts of the fluid preſs equally at the ſame depth.*

For imagine a plain drawn through the fluid parallel to the horizon. Then the preſſure will be the ſame in any part of that plane, and therefore the parts of the fluid at the ſame depth ſuſtain the ſame preſſure.

Cor. 2. *The preſſure of a fluid at any depth, is as the depth of the fluid.*

For the preſſure is as the weight, and the weight is as the height of a column of the fluid.

PROP.

Fig.

PROP. LXV.

If a fluid is compreſſed by its weight or otherwiſe; at any point it preſſes equally, in all manner of directions.

This ariſes from the nature of fluidity; which is, to yield to any force in any direction. If it cannot give way to any force applied, it will preſs againſt other parts of the fluid in direction of that force. And the preſſure in all directions will be the ſame. For if any one was leſs, the fluid would move that way, till the preſſure be equal every way.

Cor. *In any veſſel containing a fluid; the preſſure is the ſame againſt the bottom, as againſt the ſides, or even upwards, at the ſame depth.*

PROP. LXVI.

The preſſure of a fluid upon the baſe of the containing veſſel, is as the baſe, and perpendicular altitude; whatever be the figure of the veſſel that contains it.

Let ABIC, EGKH be two veſſels. Then 69.
(Prop. LXIV. Cor. 2.) the preſſure upon an inch on the baſe AB = height CD × 1 inch. And the preſſure upon an inch on the baſe HK is = height FH × 1 inch. But (Prop. LXIV.) equal parts of the baſes are equally preſſed, therefore the preſſure on the baſe AB is CD × number of inches in AB; and preſſure on the baſe HK is FH × number of inches in HK. That is, the preſſure on AB is to the preſſure on HK; as baſe AB × height CD, to the baſe HK × height FH.

Cor. 1. *Hence if the heights be equal, the preſſures are as the baſes. And if both the heights and baſes be equal;*

Fig. 69. *equal; the preſſures are equal in both; though their contents be never ſo different.*

For the reaſon that the wider veſſel EK, has no greater preſſure at the bottom, is, becauſe the oblique ſides EH, GK, take off part of the weight. And in the narrower veſſel CB, the ſides CA, IB, re-act againſt the preſſure of the water, which is all alike at the ſame depth; and by this re-action the preſſure is encreaſed at the bottom, ſo as to become the ſame every where.

Cor. 2. *The preſſure againſt the baſe of any veſſel, is the ſame as of a cylinder of an equal baſe and height.*

70. Cor. 3. *If there be a recurve tube* ABF, *in which are two different fluids* CD, EF. *Their heights in the two legs* CD, EF, *will be reciprocally as their ſpecific gravities, when they are at reſt.*

For if the fluid EF be twice or thrice as light as CD; it muſt have twice or thrice the height, to have an equal preſſure, to counterbalance the other.

PROP. LXVII.

71. *If a body of the ſame ſpecific gravity of a fluid, be immerſed in it, it will reſt in any place of it. A body of greater denſity will ſink; and one of a leſs denſity will ſwim.*

Let A, B, C be three bodies; whereof A is lighter bulk for bulk than the fluid; B is equal; and C heavier. The body B, being of the ſame denſity, or equal in weight as ſo much of the fluid, it will preſs the fluid under it juſt as much as if the ſpace was filled with the fluid. The preſſure then will be the ſame all around it, as if the fluid was there, and conſequently there is no force to put it out of its place. But if the body be lighter,

the

the preſſure of it downwards will be leſs than before, and leſs than in other places at the ſame depth; and conſequently the leſſer force will give way, and it will riſe to the top. And if the body be heavier, the preſſure downwards will be greater than before; and the greater preſſure will prevail and carry it to the bottom. Fig. 71.

Cor. 1. *Hence if ſeveral bodies of different ſpecific gravity be immerſed in a fluid, the heavieſt will get the loweſt.*

For the heavieſt are impelled with a greater force, and therefore will go faſteſt down.

Cor. 2. *A body immerſed in a fluid, loſes as much weight, as an equal quantity of the fluid weighs. And the fluid gains it.*

For if the body is of the ſame ſpecific gravity as the fluid; then it will loſe all its weight. And if it be lighter or heavier, there remains only the difference of the weights of the body and fluid, to move the body.

Cor. 3. *All bodies of equal magnitudes, loſe equal weights in the ſame fluid. And bodies of different magnitudes loſe weights proportional to the magnitudes.*

Cor. 4. *The weights loſt in different fluids, by immerging the ſame body therein, are as the ſpecific gravities of the fluids. And bodies of equal weight, loſe weights in the ſame fluid, reciprocally as the ſpecific gravities of the bodies.*

Cor. 5. *The weight of a body ſwimming in a fluid, is equal to the weight of as much of the fluid, as the immerſed part of the body takes up.*

For the preſſure underneath the ſwimming body is juſt the ſame as ſo much of the immerſed fluid; and therefore the weights are the ſame.

Cor.

Fig. 71. Cor. 6. *Hence a body will ſink deeper in a lighter fluid than in a heavier.*

Cor. 7. *Hence appears the reaſon why we do not feel the whole weight of an immerſed body, till it be drawn quite out of the water.*

PROP. LXVIII.

72. *If a fluid runs through a pipe, ſo as to leave no vacuities; the velocity of the fluid in different parts of it, will be reciprocally as the tranſverſe ſections, in theſe parts.*

Let AC, LB be the ſections at A and L. And let the part of the fluid ACBL come to the place *acbl*. Then will the ſolid ACBL = ſolid *acbl*; take away the part *ac*BL common to both; and we have AC*ca* = LB*bl*. But in equal ſolids the baſes and heights are reciprocally proportional. But if D*f* be the axis of the pipe, the heights D*d*, F*f*, paſſed through in equal times, are as the velocities. Therefore, ſection AC : ſection LB : : velocity along F*f* : velocity along D*d*.

PROP. LXIX.

73. *If* AD *is a veſſel of water or any other fluid*; B *a hole in the bottom or ſide. Then if the veſſel be always kept full; in the time a heavy body falls through half the height of the water above the hole* AB, *a cylinder of water will flow out of the hole, whoſe height is* AB, *and baſe the area of the hole.*

The preſſure of the water againſt the hole B, by which the motion is generated, is equal to the weight of a column of water whoſe height is AB, and baſe the area B (by Cor. 2. Prop. LXVI.). But equal forces generate equal motions; and ſince a cylinder

cylinder of water falling through ½AB by its gravity, acquires ſuch a motion, as to paſs through the whole height AB in that time; therefore in that time the water running out muſt acquire the ſame motion. And that the effluent water may have the ſame motion, a cylinder muſt run out whoſe length is AB; and then the ſpace deſcribed by the water in that time will alſo be AB, for that ſpace is the length of the cylinder run out. Therefore this is the quantity run out in that time. Fig. 73.

Cor. 1. *The quantity run out in any time is equal to a cylinder or priſm, whoſe length is the ſpace deſcribed in that time by the velocity acquired by falling through half the height, and whoſe baſe is the hole.*

For the length of the cylinder is as the time of running out.

Cor. 2. *The velocity a little without the hole, is greater than in the hole; and is nearly equal to the velocity of a body falling through the whole height* AB.

For without the hole the ſtream is contracted by the water's converging from all ſides to the center of the hole. And this makes the velocity greater in about the ratio of 1 to $\sqrt{2}$.

Cor. 3. *The water ſpouts out with the ſame velocity, whether it be downwards, or ſideways, or upwards. And therefore if it be upwards, it aſcends nearly to the height of the water above the hole.*

Cor. 4. *The velocities and likewiſe the quantities of the ſpouting water, at different depths; will be as the ſquare roots of the depths.*

Scholium.

From hence are derived the rules for the conſtruction of fountains or jets. Let ABC be a reſervoir of water, CDE a pipe coming from it, to 74.

Fig. 74. bring water to the fountain which ſpouts up at E, to the height EF, near to the level of the reſervoir AB. In order to have a fountain in perfection, the pipe CD muſt be wide, and covered with a thin plate at E with a hole in it, not above the fifth or ſixth part of the diameter of the pipe CD. And this pipe muſt be curve, having no angles. If the reſervoir be 50 feet high, the diameter of the hole at E may be an inch, and the diameter of the pipe 6 inches. In general, the diameter of the hole E, ought to be as the ſquare root of the height of the reſervoir. When the water runs through a great length of pipe, the jet will not riſe ſo high. A jet never riſes to the full height of the reſervoir; in a 5 feet jet it wants an inch, and it falls ſhort by lengths which are as the ſquares of the heights; and ſmaller jets loſe more. No jet will riſe 300 feet high.

75. A ſmall fountain is eaſily made by taking a ſtrong bottle A, and filling it half full of water; cement a tube BI very cloſe in it, going near the bottom of the bottle. Then blow in at the top B, to compreſs the air within; and the water will ſpout out at B. If a fountain be placed in the ſunſhine and made to play, it will ſhew all the colours of the rainbow, if a black cloth be placed beyond it.

A jet goes higher if it is not exactly perpendicular; for then the upper part of the jet falls to one ſide without reſiſting the column below. The reſiſtance of the air will alſo deſtroy a deal of its motion, and hinder it from riſing to the height of the reſervoir. Alſo the friction of the tube or pipe of conduct has a great ſhare in retarding the motion.

78. If there be an upright veſſel as AF full of water, and ſeveral holes be made in the ſide as B, C, D: then the diſtances, the water will ſpout, upon the horizontal plane EL, will be as the ſquare roots of the rectangles of the ſegments, ABE, ACE, and ADE. For the ſpaces will be as the velocities

velocities and times. But (Cor. 4.) the velocity of the water flowing out of B, will be as $\sqrt{AB}$, and the time of its moving (which is the same as the time of its fall) will be (by Prop. XIII.) as $\sqrt{BE}$; therefore the distance EH is as $\sqrt{AB \times BE}$; and the space EL as $\sqrt{ACE}$. And hence if two holes are made equidistant from top and bottom, they will project the water to the same distance, for if AB = DE, then ABE = ADE, which makes EH the same for both, and hence also it follows, that the projection from the middle point C will be furthest; for ACE is the greatest rectangle. These are the proportions of the distances; but for the absolute distances, it will be thus. The velocity through any hole B, will carry it through AB in the time of falling through $\frac{1}{2}$AB; then to find how far it will move in the time of falling through BE. Since these times are as the square roots of the heights, it will be, $\sqrt{\frac{1}{2}AB} : AB :: \sqrt{BE} : EH = AB\sqrt{\frac{BE}{\frac{1}{2}AB}} = \sqrt{2ABE}$; and so the space $EL = \sqrt{2ACE}$. It is plain, these curves are parabolas. For the horizontal motion being uniform; EH will be as the time; that is, as $\sqrt{BE}$, or BE will be as EH^2, which is the property of a parabola. Fig. 78.

If there be a broad vessel ABDC full of water, and the top AB fits exactly into it; and if the small pipe FE of a great length be soldered close into the top, and if water be poured into the top of the pipe F, till it be full; it will raise a great weight laid upon the top, with the little quantity of water contained in the pipe; which weight will be nearly equal to a column of the fluid, whose base is the top AB; and height, that of the pipe EF. For the pressure of the water against the top AB, is equal to the weight of that column of wa- 76.

Fig. 76. ter, by Prop. LXV. and Cor. And Prop. LXVI. Cor. 2.

But here the tube muſt not be too ſmall. For in capillary tubes the attraction of the glaſs will take off its gravity. If a very ſmall tube be immerſed with one end in a veſſel of water, the water will riſe in the tube above the ſurface of the water; and the higher, the ſmaller the tube is. But in quickſilver, it deſcends in the tube below the external ſurface, from the repulſion of the glaſs.

77. To explain the operation of a ſyphon, which is a crooked pipe CDE, to draw liquors off. Set the ſyphon with the ends C, E, upwards, and fill it with water at the end E till it run out at C; to prevent it, clap the finger at C, and fill the other end to the top, and ſtop that with the finger. Then keeping both ends ſtopt, invert the ſhorter end C into a veſſel of water AB, and take off the fingers, and the water will run out at E, till it be as low as C in the veſſel; provided the end E be always lower than C. Since E is always below C, the height of the column of water DE is greater than that of CD, and therefore DE muſt out weigh CD and deſcend, and CD will follow after, being forced up by the preſſure of the air, which acts upon the ſurface of the water in the veſſel AB.

The ſurface of the earth falls below the horizontal level only an inch in 620 yards; and in other diſtances the deſcents are as the ſquares of the diſtances.

79. And to find the nature of the curve DCG, forming the jet IDG. Let AK be the height or top of the reſervoir HF, and ſuppoſe the ſtream to aſcend without any friction, or reſiſtance. By the laws of falling bodies the velocity in any place B, will be as $\sqrt{AB}$. Put the ſemidiameter of the hole at D $= d$, and AD $= b$. Then ſince the ſame water paſſes through the ſections at D and B; therefore (Prop. LXVIII.) the velocity will be reciprocally

as

as the ſection; whence $\sqrt{h} : \frac{1}{dd} :: \sqrt{AB} : \frac{1}{BC^2}$; Fig. 79.

therefore $\frac{\sqrt{h}}{BC^2} = \frac{\sqrt{AB}}{dd}$, and $dd\sqrt{h} = BC^2\sqrt{AB}$,

whence $AB \times BC^4 = hd^4$; which is a paraboliform figure whoſe aſymptote is AK, for the nature of the cataractic curve DCG. And if the fluid was to deſcend through a hole, as IC; it would form itſelf into the ſame figure GCD in deſcending.

PROP. LXX.

The reſiſtance any body meets with in moving through a fluid is as the ſquare of the velocity.

For if any body moves with twice the velocity of another body equal to it, it will ſtrike againſt twice as much of the fluid, and with twice the velocity; and therefore has four times the reſiſtance; for that will be as the matter and velocity. And if it moves with thrice the velocity, it ſtrikes againſt thrice as much of the fluid in the ſame time, with thrice the velocity, and therefore has nine times the reſiſtance. And ſo on for all other velocities.

Cor. *If a ſtream of water whoſe diameter is given, ſtrike againſt an obſtacle at reſt; the force againſt it will be as the ſquare of the velocity of the ſtream.*

For the reaſon is the ſame; ſince with twice or thrice the velocity, twice or thrice as much of the fluid impinges upon it, in the ſame time.

PROP. LXXI.

The force of a ſtream of water againſt any plane obſtacle at reſt, is equal to the weight of a column of water, whoſe baſe is the ſection of the ſtream; and height, the ſpace deſcended through by a falling body, to acquire that velocity.

For let there be a reſervoir whoſe height is that ſpace fallen through. Then the water (by Cor. 2.

Fig. Prop. LXIX.) flowing out at the bottom of the reſervatory, has the ſame motion as the ſtream; but this is generated by the weight of that column of water, which is the force producing it. And that ſame motion is deſtroyed by the obſtacle, therefore the force againſt it is the very ſame: for there is required as much force to deſtroy as to generate any motion.

Cor. *The force of a ſtream of water flowing out at a hole in the bottom of a reſervatory, is equal to the weight of a column of the fluid of the ſame height and whoſe baſe is the hole.*

PROP. LXXII. *Prob.*

To find the ſpecific gravity of ſolids or fluids.

1. *For a ſolid heavier than water.*

Weigh the body ſeparately, firſt out of water, and then ſuſpended in water. And divide the weight out of water by the difference of the weights, gives the ſpecific gravity; reckoning the ſpecific gravity of water 1.

For the difference of the weights is equal to the weight of as much water (by Cor. 2. Prop. LXVII.); and the weights of equal magnitudes, are as the ſpecific gravities; therefore the difference of theſe weights, is to the weight of the body, as the ſpecific gravity of water 1, to the ſpecific gravity of the body.

2. *For a body lighter than water.*

Take a piece of any heavy body, ſo big as being tied to the light body, it may ſink it in water. Weigh the heavy body in and out of water, and find the loſs of weight. Alſo weigh the compound both in and out of water, and find alſo the loſs of weight.

weight. Then divide the weight of the light body (out of water), by the difference of theſe loſſes, gives the ſpecific gravity; the ſpecific gravity of water being 1. Fig.

For the laſt loſs is = weight of water equal in magnitude to the compound.

And the firſt loſs is = weight of water equal in magnitude to the heavy body.

Whence the diff. loſſes is = weight of water equal in magnitude to the light body.

and the weights of equal magnitudes, being as the ſpecific gravities; therefore the difference of the loſſes, (or the weight of water equal to the light body) : weight of the light body : : ſpecific gravity of water 1 : ſpecific gravity of the light body.

3. *For a fluid of any ſort.*

Take a piece of a body whoſe ſpecific gravity you know; weigh it both in and out of the fluid; take the difference of the weights, and multiply it by the ſpecific gravity of the ſolid body, and divide the product by the weight of the body (out of water), for the ſpecific gravity of the fluid.

For the difference of the weights in and out of water, is the weight of ſo much of the fluid as equals the magnitude of the body. And the weight of equal magnitudes being as the ſpecific gravities; therefore, weight of the ſolid : difference of the weights (or the weight of ſo much of the fluid) : : ſpecific gravity of the ſolid : to the ſpecific gravity of the fluid.

Example to Caſe 1.

I weighed a piece of lead ore, which was 124 grains; and in water it weighed 104 grains, the

Fig. difference is 20; then $\frac{124}{20} = 6.2$, the ſpecific gravity of the ore.

A table of ſpecific gravities.

Fine gold	19.640
Standard gold	18.888
Quickſilver	14.000
Lead	11.340
Fine ſilver	11.092
Standard Silver	10.536
Copper	9.000
Copper half-pence	8.915
Gun metal	8.784
Fine braſs	8.350
Caſt braſs	8.100
Steel	7.850
Iron	7.644
Pewter	7.471
Tin	7.320
Caſt iron	7.000
Lead ore	6.200
Copper ore	5.167
Lapis calaminaris	5.000
Load ſtone	4.930
Antimony	4.000
Diamond	3.517
Iſland cryſtal	2.720
Stone, hard	2.700
Rock cryſtal	2.650
Glaſs	2.600
Flint	2.570
Common ſtone	2.500
Cryſtal	2.210
Brick	2.000
Earth	1.984
Horn	1.840

Ivory

		Fig.
Ivory	1.820	
Chalk	1.793	
Allum	1.714	
Clay	1.712	
Oil of vitriol	1.700	
Honey	1.450	
Lignum vitæ	1.327	
Treacle	1.290	
Pitch	1.150	
Roſin	1.100	
Mahogany	1.063	
Amber	1.040	
Urine	1.032	
Milk	1.031	
Brazil	1.031	
Box	1.030	
Sea water	1.030	
Ale	1.028	
Vinegar	1.026	
Tar	1.015	
Common clear water	1.000	
Bees wax	.955	
Butter	.940	
Linſeed oil	.932	
Brandy	.927	
Sallad oil	.913	
Logwood	.913	
Ice	.908	
Oak	.830	
Aſh	.830	
Elm	.820	
Oil of turpentine	.810	
Walnut tree	.650	
Fir	.580	
Cork	.238	
New fallen ſnow	.086	
Air	.0012	

Cor.

Fig. Cor. 1. *As the weight loſt in a fluid, is to the abſolute weight of the body; ſo is the ſpecific gravity of the fluid, to the ſpecific gravity of the body.*

Cor. 2. *Having the ſpecific gravity of a body, and the weight of it; the ſolidity may be found thus; multiply the weight in pounds by* $62\frac{1}{2}$. *Then ſay as that product to* 1; *ſo is the weight of the body in pounds, to the content in feet. And having the content given, one may find the weight, by working backwards.*

For a cubic foot of water weighs $62\frac{1}{2}$lb. averdupois; and therefore a cubic foot of the body weighs $62\frac{1}{2}\times$ by the ſpecific gravity of the body. Whence the weight of the body, divided by that product, gives the number of feet in it. Or as 1, to that product; ſo is the content, to the weight.

Scholium.

80. The ſpecific gravities of bodies may be found with a pair of ſcales; ſuſpending the body in water, by a horſe hair. But there is an inſtrument for this purpoſe called the *Hydroſtatical Balance*, the conſtruction of which is thus. AB is the ſtand and pedeſtal, having at the top two cheeks of ſteel, on which the beam CD is ſuſpended, which is like the beam of a pair of ſcales, and muſt play freely, and be itſelf exactly in equilibrio. To this belongs the glaſs bubble G, and the glaſs bucket H, and four other parts E, F, I, L. To theſe are loops faſtened to hang them by. And the weights of all theſe are ſo adjuſted, that E = F + the bubble in water, or = I + the bucket out of water, or = I + L + the bucket in water. Whence L = difference of the weights of the bucket in and out of water. And if you pleaſe you may have a weight K, ſo that K + bubble in water = bubble out of water; or elſe find it in

grains.

grains. The piece L has a ſlit in it to ſlip it upon the ſhank of I. Fig. 80.

It is plain the weight K = weight of water as big as the bubble, or a water bubble.

Then to find the ſpecific gravity of a ſolid.

Hang E at one end of the balance, and I and the bucket with the ſolid in it, at the other end; and find what weight is a balance to it.

Then ſlip L upon I, and immerge the bucket and ſolid in the water, and find again what weight balances it. Then the firſt weight divided by the difference of the weights, is the ſpecific gravity of the body; that of water being 1.

For fluids.

Hang E at one end, and F with the bubble at the other; plunge the bubble into the fluid in the veſſel MN. Then find the weight P which makes a balance. Then the ſpecific gravity of the fluid is $= \frac{K+P}{K}$, when P is laid on F; or $= \frac{K-P}{K}$, when P is laid on E.

For E being equal to I + the bucket; the firſt weight found for a balance, is the weight of the ſolid. Again, E being equal to I + L + the bucket in water; the weight to balance that, is the weight of the ſolid in water; and the difference, is = to the weight of as much water. Therefore (Cor. 1.) the firſt weight divided by that difference, is the ſpecific gravity of the body.

Again, ſince E is = to F + the bubble in water; therefore P is the difference of the weights, of the fluid and ſo much water; that is, P = difference of K and a fluid bubble; or P = fluid − K, when the fluid is heavier than water, or when P is laid on F. And therefore P = K − the fluid bubble,

Fig. 80. bubble, when contrary. Whence the fluid bubble $= K \pm P$, for a heavier or a lighter fluid. And the ſpecific gravities being as the weights of theſe equal bubbles; ſpecific gravity of water : ſpecific gravity of the fluid : : K : $K \pm P$: : 1 : $\frac{K \pm P}{K}$ the ſpecific gravity of the fluid. Where if P be 0, it is the ſame as that of water.

PROP. LXXIII.

The air is a heavy body, and gravitates on all parts of the ſurface of the earth.

That the air is a fluid is very plain, as it yields to any the leaſt force that is impreſſed upon it, without making any ſenſible reſiſtance. But if it be moved briſkly, by ſome very thin and light body, as a fan, or by a pair of bellows, we become very ſenſible of its motion againſt our hands or face, and likewiſe by its impelling or blowing away any light bodies, that lie in the way of its motion. Therefore the air being capable of moving other bodies by its impulſe, muſt itſelf be a body; and muſt therefore be heavy like all other bodies, in proportion to the matter it contains; and will conſequently preſs upon all bodies placed under it. And being a fluid, it will dilate and ſpread itſelf all over upon the earth: and like other fluids will gravitate upon, and preſs every where upon its ſurface. The gravity and preſſure of the air is alſo evident from experiments. For (fig. 70.) if water, &c. be put into the tube ABF, and the air be drawn out of the end F by an air-pump, the water will aſcend in the end F, and deſcend in the end A, by reaſon of the preſſure at A, which was taken off or diminiſhed at F. There are numberleſs experiments of this ſort. And though theſe properties

perties and effects are certain, yet the air is a fluid ſo very fine and ſubtle, as to be perfectly tranſparent, and quite inviſible to the eye. Fig. 70.

Cor. 1. *The air, like other fluids, will, by its weight and fluidity, inſinuate itſelf into all the cavities, and corners within the earth; and there preſs with ſo much greater force, as the places are deeper.*

Cor. 2. *Hence the atmoſphere, or the whole body of air ſurrounding the earth, gravitates upon the ſurfaces of all other bodies, whether ſolid or fluid, and that with a force proportional to its weight or quantity of matter.*

For this property it muſt have in common with all other fluids.

Cor. 3. *Hence the preſſure, at any depth of water, or other fluid, will be equal to the preſſure of the fluid together with the preſſure of the atmoſphere.*

Cor. 4. *Likewiſe all bodies, near the ſurface of the earth, loſe ſo much of their weight, as the ſame bulk of ſo much air weighs. And conſequently, they are ſomething lighter than they would be in a vacuum. But being ſo very ſmall it is commonly neglected; though in ſtrictneſs, the true or abſolute weight is the weight in vacuo.*

PROP. LXXIV.

The air is an elaſtic fluid, or ſuch a one, as is capable of being condenſed or expanded. And it obſerves this law, that its denſity is proportional to the force that compreſſes it.

Theſe properties of the air, are proved by experiments, of which there are innumerable. If you take a ſyringe, and thruſt the handle inwards, you will feel the included air act ſtrongly againſt your hand;

Fig. hand; and the more you thruſt, the further the piſton goes in, but the more it reſiſts; and taking away your hand, the handle returns back to where it was at firſt. This proves its elaſticity, and alſo that air may be driven into a leſs ſpace, and condenſed.

75. Again, take a ſtrong bottle, and fill it half full of water, and cement a pipe BI, cloſe in it, going near the bottom; then inject air into the bottle through the pipe BI. Then the water will ſpout out at B, and form a jet; which proves, that the air is firſt condenſed, and then by its ſpring drives out the water, till it become of the ſame denſity as at firſt, and then the ſpouting ceaſes.

81. Likewiſe if a veſſel of glaſs AB be filled with water in the veſſel CD, and then drawn up with the bottom upwards; if any air is left in the top at A, the higher you pull it up, the more it expands; and the further the glaſs is thruſt down into the veſſel CD, the more the air is condenſed.

82. Again, take a crooked glaſs tube ABD open at the end A, and cloſe at D; pour in mercury to the height BC, but no higher, and then the air in DC is in the ſame ſtate as the external air. Then pour in more mercury at A, and obſerve where it riſes to in both legs, as to G and H. Then you may always ſee that the higher the mercury is in the leg BH, the leſs the ſpace GD is, into which the air is driven. And if the height of the mercury FH be ſuch as to equal the preſſure of the atmoſphere, then DG will be half DC; if it be twice the preſſure of the atmoſphere, DG will be $\frac{1}{3}$DC, &c. So that the denſity is always as the weight or compreſſion. And here the part CD is ſuppoſed to be cylindrical.

Cor. 1. *The ſpace that any quantity of air takes up, is reciprocally as the force that compreſſes it.*

Cor.

Cor. 2. *All the air near the earth is in a ſtate of compreſſion by the weight of the incumbent atmoſphere.* Fig.

Cor. 3. *The air is denſer near the earth, or at the foot of a mountain, than at the top of it, and in high places. And the higher from the earth, the more rare it is.*

Cor. 4. *The ſpring or elaſticity of the air is equal to the weight of the atmoſphere above it; and produces the ſame effects.*

For they always balance and ſuſtain each other..

Cor. 5. *Hence if the denſity of the air be increaſed; its ſpring or elaſticity will likewiſe be encreaſed in the ſame proportion.*

Cor. 6. *From the gravity and preſſure of the atmoſphere, upon the ſurfaces of fluids, the fluids are made to riſe in pipes or veſſels, when the preſſure within is taken off.*

PROP. LXXV.

The expanſion and elaſticity of the air is increaſed by heat, and decreaſed by cold. Or heat expands, and cold condenſes the air.

This is alſo a matter of experience; for tie a bladder very cloſe with ſome air in it, and lay it before the fire, and it will viſibly diſtend the bladder; and burſt it if the heat is continued, and encreaſed high enough.

If a glaſs veſſel AB (fig. 81) with water in it, 81.
be turned upſide down, with a little air in the top A; and be placed in a veſſel of water, and hung over the fire, and any weight laid upon it to keep it down; as the water warms, the air in the top A, will by degrees expand, till it fills the glaſs, and by

Fig. 81. by its elaſtic force, drive all the water out of the glaſs, and a good part of the air will follow, by continuing the veſſel there. Many more experiments may be produced proving the ſame thing.

PROP. LXXVI.

The air will preſs upon the ſurfaces of all fluids, with any force; without paſſing through them, or entering into them.

If this was not ſo, no machine, whoſe uſe or actions depends upon the preſſure of the atmoſphere, could do its buſineſs. Thus the weight of the atmoſphere preſſes upon the ſurface of water, and forces it up into the barrel of a pump, without any air getting in, which would ſpoil its working. Likewiſe the preſſure of the atmoſphere keeps mercury ſuſpended at ſuch a height, that its weight is equal to that preſſure; and yet it never forces itſelf through the mercury into the vacuum above, though it ſtand never ſo long. And whatever be the texture or conſtitution of that ſubtle inviſible fluid we call air, yet it is never found to paſs through any fluid, though it be made to preſs never ſo ſtrongly upon it. For though there be ſome air incloſed in the pores of almoſt all bodies, whether ſolid or fluid; yet the particles of air cannot by any force be made to paſs through the body of any fluid; or forced through the pores of it, although that force or preſſure be continued never ſo long. And this ſeems to argue that the particles of air are greater than the particles or pores of other fluids; or at leaſt are of a ſtructure quite different from any of them.

Fig.

PROP. LXXVII.

The weight or preſſure of the atmoſphere, upon any baſe at the earth's ſurface; is equal to the weight of a column of mercury of the ſame baſe, and whoſe height is from 28 *to* 31 *inches; ſeldom more or leſs.*

This is evident from the barometer, an inſtrument which ſhews the preſſure of the air; which at ſome ſeaſons ſtands at a height of 28 inches, ſometimes at 29, and 30, or 31. The reaſon of this is not becauſe there is at ſome times more air in the atmoſphere, than at others; but becauſe the air being an extremely ſubtle and elaſtic fluid, capable of being moved by any impreſſions, and many miles high; it is much diſturbed by winds, and by heat and cold; and being often in a tumultuous agitation; it happens to be accumulated in ſome places, and conſequently depreſſed in others; by which means it becomes denſer and heavier where it is higher, ſo as to raiſe the column of mercury to 30 or 31 inches. And where it is lower, it is rarer and lighter, ſo as only to raiſe it to 28 or 29 inches. And experience ſhews, that it ſeldom goes without the limits of 28 and 31.

Cor. 1. *The air in the ſame place does not always continue of the ſame weight; but is ſometimes heavier, and ſometimes lighter; but the mean weight of the atmoſphere, is that when the quickſilver ſtands at about* $29\frac{1}{2}$ *inches.*

Cor. 2. *Hence the preſſure of the atmoſphere upon a ſquare inch at the earth's ſurface, at a medium, is very near* 15 *pounds, averdupoiſe.*

For an inch of quickſilver weighs 8.102 ounces.

Cor. 3. *Hence alſo the weight or preſsure of the atmoſphere, in its lighteſt and heavieſt ſtate, is equal to*

Fig. *the weight of a column of water, 32 or 36 feet high; or at a medium 34 feet.*

For water and quickſilver are in weight nearly as 1 to 14.

Cor. 4. *If the air was of the ſame denſity to the top of the atmoſphere, as it is at the earth; its height would be about* $5\frac{1}{3}$ *miles at a medium.*

For the weight of air and water are nearly as 12 to 10000.

Cor. 5. *The denſity of the air in two places diſtant from each other but a few miles, on the earth's ſurface and in the ſame level; may be looked on to be the ſame, at the ſame time.*

Cor. 6. *The denſity of the air at two different altitudes in the ſame place, differing only by a few feet; may be looked on as the ſame.*

Cor. 7. *If the perpendicular height of the top of a ſyphon from the water, be more than 34 feet, at a mean denſity of the air; the ſyphon cannot be made to run.*

For the weight of the water in the legs will be greater than the preſſure of the atmoſphere, and both columns will run down, till they be 34 feet high.

Cor 8. *Hence alſo the quickſilver riſes higher in the barometer, at the bottom of a mountain, than at the top; and at the bottom of a coal pit, than at the top of it.*

SCHOLIUM.

Fig.

Scholium.

Hence the denſity of the air may be found at any height from the earth as in the following table.

Miles	denſity	Miles	denſity
$\frac{1}{4}$	.9564	10	.1700
$\frac{1}{2}$	.9146	20	.02917
$\frac{3}{4}$	.8748	30	.005048
1	.8372	40	.000881
2	.7012	50	.000155
3	.5871	100	.0000000298
4	.4917		
5	.4119		

The firſt and third columns are the height in miles from the ſurface of the earth. And the ſecond and fourth columns, ſhew the denſity at that height; ſuppoſing the denſity at the ſurface of the earth, to be 1.

The denſity at any height is eaſily calculated by this ſeries. Put r = radius of the earth, h = height from the ſurface, both in feet. Then the denſity at the height h, is the number belonging to the logarithm, denoted by this ſeries $-\frac{h}{68444} - \frac{h}{r}A - \frac{h}{r}B - \frac{h}{r}C$ &c. where A, B, C, &c. are the preceding terms. The terms here will be alternately negative and affirmative. But the firſt term alone is ſufficient when the height is but a few miles.

By the weight and preſſure of the atmoſphere, the operations of pneumatic engines may be accounted for and explained. I ſhall juſt mention one or two.

A PUMP.

Fig. 83. is a *common pump*. AB the barrel or body of the pump, being a hollow cylinder, made 83.

Fig. 83. of wood or lead. CD the handle movable about the pin E. DF an iron rod moving about a pin D; this rod is hooked to the bucket or ſucker FG, which moves up and down within the pump. The bucket FG is hollow, and has a valve or clack L at the top opening upwards. H a plug fixed at the bottom of the barrel, being likewiſe hollow, and a valve at I opening alſo upwards. BK the bottom going into the well at K; the pipe below B need not be large, being only to convey the water out of the well into the body of the pump. The plug H muſt be fixed cloſe that no water can get between it and the barrel; and the ſucker FG, is to be armed with leather, to fit cloſe that no air or water can get through between it and the barrel.

When the pump is firſt wrought, or any time in dry weather when the water above the ſucker is waſted, it muſt be primed, by pouring in ſome water at the top A to cover the ſucker, that no air get through. Then raiſing the end C of the handle, the bucket F deſcends, and the water will riſe through the hollow GL, preſſing open the valve L. Then putting down the end C raiſes the bucket F, and the valve L ſhuts by the weight of the water above it. And at the ſame time the preſſure of the atmoſphere forces the water up through the pipe KB, and opening the valve I, it paſſes through the plug into the body of the pump. And when the ſucker G deſcends again, the valve I ſhuts, and the water cannot return, but opening the valve L, paſſes through the ſucker GL. And when the ſucker is raiſed again, the valve L ſhuts again, and the water is raiſed in the pump. So that by the motion of the piſton up and down, and the alternate opening and ſhutting of the two valves; water is continually raiſed into the body of the pump, and diſcharged at the ſpout M.

The

The diſtance KG, from the well to the bucket, Fig. muſt not be above 32 feet; for the preſſure of the 83. atmoſphere will raiſe the water no higher, and if it is more, the pump will not work. It is evident a pump will work better when the atmoſphere is heavy than when it is light, there being a twelfth or fifteenth part difference, at different times. And when it is lighteſt it is only equal to 32 feet. Wherefore the plug H muſt always be placed ſo low, as that the ſucker GL may be within that compaſs.

A BAROMETER.

Fig. 84. is a *Barometer*, or an inſtrument to mea- 84. ſure the weight of the air. It conſiſts of a glaſs cone ABC hollow within, filled full of mercury, and hermetically ſealed at the end C, ſo that no air be left in it. When it is ſet upright, the mercury deſcends down the tube BC, into the bubble A, which has a little opening at the top A, that the air may have free ingreſs and egreſs. At the top of the tube C, there muſt be a perfect vacuum. This is fixed in a frame, and hung perpendicular againſt a wall. Near the top C, on the frame, is placed a ſcale of inches, ſhewing how high the mercury is in the tube BC, above the level of it in the bubble A, which is generally from 28 to 31 inches, but moſtly about 29 or 30. Along with the ſcale of inches, there is alſo placed a ſcale of ſuch weather as has been obſerved to anſwer the ſeveral heights of the quickſilver. Such a ſcale you have annexed to the 84th figure. In dividing the ſcale of inches, care muſt be taken to make proper allowance for the riſing or falling of the quickſilver in the bubble A, which ought to be about half full, when it ſtands $29\frac{1}{2}$, which is the mean height. For whilſt the quickſilver riſes an inch at C, it deſcends a little in the bubble A, and that

Fig. 84. deſcent muſt be deducted, which makes the diviſions be ſomething leſs than an inch. Theſe inches muſt be divided into tenth parts, for the more exact meaſuring the weight of the atmoſphere. For the pillar of mercury in the tube is always equal to the weight of a pillar of the atmoſphere of the ſame thickneſs. And as the height of the quickſilver increaſes or decreaſes the weight of the air increaſes or decreaſes accordingly. The tube muſt be near 3 feet long, and the bore not leſs than $\frac{1}{5}$ or $\frac{1}{6}$ of an inch, in diameter, or elſe the quickſilver will not move freely in it.

By help of the barometer, the height of mountains may be meaſured by the following table. In which the firſt column is the height of the mountain, &c. in feet or miles; the ſecond the height of the quickſilver; and the third the deſcent of the quickſilver in the barometer; and this at a mean denſity of the air.

Feet

Fig.

Feet	High Barom.	Descent	Feet	High Barom	Descent
0	29.500				
100	29.400	.100	2600	27.028	2.472
200	29.301	.199	2700	26.938	2.562
300	29.203	.297	2800	26.848	2.652
400	29.105	.395	2900	26.758	2.742
500	29.007	.493	3000	26.668	2.832
600	28.910	.590	3100	26.578	2.922
700	28.812	.688	3200	26.489	3.011
800	28.716	.784	3300	26.400	3.100
900	28.619	.881	3400	26.311	3.189
1000	28.523	.977	3500	26.222	3.278
1100	28.428	1.072	3600	26.136	3.364
1200	28.332	1.168	3700	26.049	3.451
1300	28.237	1.263	3800	25.961	3.539
1400	28.143	1.357	3900	25.874	3.626
1500	28.048	1.452	4000	25.786	3.714
1600	27.954	1.546	4100	25.699	3.801
1700	27.860	1.640	4200	25.613	3.887
1800	27.766	1.734	4300	25.527	3.973
1900	27.672	1.828	4400	25.441	4.059
2000	27.579	1.921	4500	25.355	4.145
2100	27.487	2.013	4600	25.270	4.230
2200	27.394	2.106	4700	25.185	4.315
2300	27.302	2.198	4800	25.101	4.399
2400	27.210	2.290	4900	25.017	4.483
2500	27.119	2.381	5000	24.933	4.567

Fig. 84.

The Table continued in MILES.

Miles	H. Barom.	Deſcent	Miles	H. Barom.	Deſcent
0.	29.50				
0.25	28.21	1.29	3.25	16.57	12.93
0.50	26.98	2.52	3.50	15.85	13.65
0.75	25.80	3.70	3.75	15.16	14.34
1.	24.70	4.80	4.	14.50	15.00
1.25	23.62	5.88	4.25	13.87	15.63
1.50	22.60	6.90	4.50	13.27	16.23
1.75	21.62	7.88	4.75	12.70	16.80
2.	20.68	8.82	5.	12.15	17.35
2.25	19.78	9.72	5.25	11.62	17.88
2.50	18.93	10.57	5.50	11.12	18.38
2.75	18.11	11.39	5.75	10.64	18.86
3.	17.32	12.18	6.	10.18	19.32

This table is made from a table of the air's denſity, made as in Schol. Prop. LXXVII. And then multiplying all the numbers thereof by 29.5 the mean denſity of the air. For the denſity of the air at any height above the earth is as the weight of the atmoſphere above it, (by Prop. LXXIV.); and that is as the height of the mercury in the barometer.

I lately contrived another ſort of barometer, which ſhews the aſcent and deſcent of the mercury at the bottom.

85. ABC is a recurve tube, cloſe at the top, where the bucket C is, and open at the end A. The length of CB is 32 or 33 inches, and of AB 6 or 7. The bucket C ſhould contain about as much as the end AB. And the bucket and end CB muſt be quite filled with mercury, as far as B, a little beyond the turn. The wider the bucket C is, the better. The ſcale ſet to the end AB muſt be graduated downwards, for the mercury falls in this

this, when it riſes in the other ſort. This being placed againſt a wall, will ſhew the height of the mercury, as in the common ones. And this way is more commodious, as it ſaves the labour of clambering up upon chairs to ſee it, as one muſt do, in the common ſort, to ſee exactly. Fig. 85.

A WATER BAROMETER.

A barometer may alſo be made of water as in fig. 86, which is a water barometer. AB is a glaſs tube open at both ends, and cemented cloſe in the mouth of the bottle EF, and reaching very near the bottom. Then warming the bottle at the fire, part of the air will fly out; then the end A is put into a veſſel of water mixed with cochineal, which will go through the pipe into the bottle as it grows cold. Then it is ſet upright; and the water may be made to ſtand at any point C, by ſucking or blowing at A. And if this barometer be kept to the ſame degree of heat, by putting it in a veſſel of ſand, it will be very correct for taking ſmall altitudes; for a little alteration in the weight of the atmoſphere, will make the water at C riſe or fall in the tube very ſenſibly. But if it be ſuffered to grow warmer, the water will riſe too high in the tube, and ſpoil the uſe of it; ſo that it muſt be kept to the ſame temper. 86.

If a barometer was to be made of water put into an exhauſted tube, after the manner of quickſilver; it would require a tube 36 feet long or more; which could hardly find room within doors. But then it would go 14 times more exact than quickſilver; becauſe for every inch the quickſilver riſes, the water would riſe 14; from whence every minute change in the atmoſphere would be diſcernable.

And the water barometer above deſcribed will ſhew the variation of the air's gravity as minutely

Fig. 86. as the other, if the bottle be large to hold a great quantity of air. And in any caſe, by reducing the botttle (ſo far as the air is contained) to a cylinder; and put D = diameter of the bottle, d = diameter of the pipe, p = height of air, x = riſing in the pipe, all in inches. Then the height of a hill in feet will be nearly $1 + \sqrt{\frac{408dd}{pDD}} \times 71x$. And if y = height of the hill or any aſcent, $Q = \frac{408dd}{pDD}$. Then $x = \frac{y}{1 + \sqrt{Q} \times 71}$ very near, at a mean denſity of the air.

A THERMOMETER.

87. Fig. 87. is a *thermometer*, or an inſtrument to meaſure the degrees of heat and cold. AB is a hollow tube near two foot long, with a ball at the bottom; it is filled with ſpirits of wine mixed with cochineal, half way up the neck; which done, it is heated very much, till the liquor fill the tube, and then it is ſealed hermetically at the end A. Then the ſpirit contracts within the tube as it cools. It is incloſed in a frame, which is graduated into degrees, for heat and cold. For hot weather dilates the ſpirit, and makes it run further up the tube; and cold weather on the contrary, contracts it, and makes it ſink lower in the tube. And the particular diviſions, ſhew the ſeveral degrees of heat and cold; againſt the principal of which, the words heat, cold, temperate, &c. are written.

They that would ſee more machines deſcribed, may conſult my large book of Mechanics, where he will meet with great variety.

THE PROJECTION OF THE SPHERE,

ORTHOGRAPHIC, STEREOGRAPHIC, and GNOMONICAL.

Both demonſtrating the

PRINCIPLES,

And explaining the

PRACTICE

Of theſe three ſeveral Sorts of PROJECTION.

In Minimis Uſus——

PREFACE.

THE *Projection of the Sphere, or of its Circles, has the ſame relation to Spherical Trigonometry, that practical Geometry has to plane Trigonometry. For as the one ſaves a deal of Calculation, by drawing a few right Lines, ſo does the other by drawing a few Circles. The Projection of the Sphere gives a Learner a good Idea of the Sphere and all its Circles, and of their ſeveral Poſitions to one another, and conſequently of Spherical Triangles, and the Nature of Spherical Trigonometry.*

I have here delivered the Principles of three ſorts of Projection, in a ſmall compaſs; and yet the Reader will find here, all that is eſſential to the ſubject; and yet nothing ſuperfluous; for I think no more need be ſaid, or indeed can be ſaid about it, to make it intelligible and practicable. For here is laid down, not only the whole Theory, but the Practice likewiſe. Yet the practical Part is entirely diſengaged from the Theory; ſo that any body (though he has no deſire or leiſure to attain to the Theory) may nevertheleſs, by help of the Problems, make himſelf Maſter of the Practice. For which end I have endeavoured to make all the rules relating to practice, plain, ſhort and eaſy, and at the ſame time full and clear.

It is true the ſolution of Problems this way, muſt be allowed to be imperfect; for there will always be ſome errors in working, as well as in the inſtruments we work with. But nobody

in

in ſeeking an accurate ſolution to a Problem, will truſt to a Projection by ſcale and compaſs; becauſe this cannot be depended on in caſes of great nicety. Yet where no great exactneſs is required, it will be found very ready and uſeful; and beſides it will ſerve to prove and confirm the ſolution obtained by Calculation.

But then this defect is abundantly recompenſed by the eaſineſs of this method. For by ſcale and compaſs only, all ſorts of Problems belonging to the Sphere, as in Aſtronomy, Geography, Dialing, &c. may be ſolved with very little trouble, which require a great deal of time and pains, to work out trigonometrically by the tables. It likewiſe affords a great pleaſure to the mind, that one can, in a little time, deſcribe the whole furniture of Heaven and Earth, and repreſent them to the eye, in a ſmall ſcheme of paper.

But its principal uſe is for ſuch perſons (and that is by far the greater number) as having no opportunity for learning Spherical Trigonometry, have yet a deſire to reſolve ſome Problems of the Sphere. For ſuch as theſe, this ſmall Treatiſe will be of particular ſervice, becauſe the practical rules, eſpecially of any one ſort of Projection, may be learned in a very little time, and are eaſily remembered. So that I have ſome hopes I ſhall pleaſe all my readers, whether theoretical or practical.

W. E.

THE PROJECTION OF THE SPHERE IN PLANO.

DEFINITIONS.

1. *PROJECTION* of the ſphere is the repreſenting its ſurface upon a plane, called the *Plane of Projection.*

2. *Orthographic* Projection, is the drawing the circles of the ſphere upon the plane of ſome great circle, by lines perpendicular to that plane, let fall from all the points of the circles to be projected.

3. The *Stereographic* Projection, is the drawing the circles of the ſphere upon the plane of one of its great circles, by lines drawn from the pole of that great circle to all the points of the circles to be projected.

4. The *Gnomonical* Projection, is the drawing the circles of an hemiſphere, upon a plane touching it in the vertex, by lines or rays iſſuing from the center of the hemiſphere, to all the points of the circles to be projected.

5. The *Primitive* circle is that on whoſe plane the ſphere is projected. And the pole of this circle is called the *Pole* of Projection. The point from whence the *projecting* right lines iſſue, is the *projecting Point.*

6. The

6. The *Line of Meaſures* of any circle is the common interſection of the plane of projection, and another plane that paſſes through the eye, and is perpendicular both to the plane of projection, and to the plane of that circle.

Scholium.

There are other Projections of the Sphere, as the *Cylindrical*, the *Scenographhic* which belongs to Perſpective, the *Globical* which belongs to Geography, *Mercator's*, for which ſee Navigation, &c.

AXIOM.

The *Place* of any viſible point of the Sphere upon the plane of projection, is where the projecting line cuts that plane.

Cor. *If the eye be applied to the projecting point, it will view all the circles of the Sphere, and every part of them, in the projection, juſt as they appear from thence in the Sphere itſelf.*

Scholium.

The Projection of the Sphere is only the ſhadow of the circles of the Sphere upon the plane of Projection, the light being in the place of the eye or projecting point.

The Signification of ſome Characters.

+ added to.
− ſubtracting the following quantity.
∠ an angle.
= equal to.
⊥ perpendicular to.
|| parallel to.
: : a proportion.

SECT. I.

The Orthographic Projection of the SPHERE.

PROP. I.

IF a right line AB *is projected upon a plane, it is projected into a right line; and its length will be to the length of the projection, as radius to the cosine of its inclination above that plane.* Fig. 3.

For let fall the perpendiculars A*a*, B*b* upon the plane of projection; then *ab* will be the line it is projected into; but by trigonometry AB : is to A*o* or *ab* : : as radius : to the sine of B or cosine of *o*AB.

Cor 1. *If a right line is projected upon a plane, parallel thereto, it is projected into a right line parallel and equal to itself.*

Cor. 2. *If an angle be projected upon a plane which is parallel to the two lines forming the angle; it is projected into an angle equal to itself.*

Cor. 3. *Any plane figure projected upon a plane parallel to itself, is projected into a figure similar and equal to itself.*

Cor. 4. *Hence also the area of any plane figure, is to the area of its projection : : as radius, to the cosine of its elevation or inclination.*

PROP. II.

A circle perpendicular to the plane of projection, is projected into a right line equal to its diameter.

For projecting lines drawn through all the points of the circle fall in the common section of the planes

Fig. of the circle and of projection, which is a right line (Geom. V. 3.), and equal to the diameter of the circle; becauſe the planes interſect in that diameter. *Q. E. D.*

Cor. *Hence any plane figure, perpendicular to the plane of projection is projected into a right line. For the perpendiculars from every point, will all fall in the common interſection of the figure with the plane of projection.*

PROP. III.

1. *A circle parallel to the plane of projection is projected into a circle equal to itſelf, and concentric with the primitive.*

Let BOD be the circle, I its center, C the center of the ſphere; the points I, B, O, D, are projected into the points C, L, F, G, and therefore OICF, and BICL are rectangled parallelograms. Conſequently LC = BI = OI = FC, (Geom. III. 1.). *Q. E. D.*

Cor. *The radius* CL *or* CF *is the coſine of the circle's diſtance from the primitive, for it is the ſine of* AB.

PROP. IV.

2. *An inclined circle is projected into an ellipſis whoſe tranſverſe axis is the diameter of the circle.*

Let ADBH be the inclined circle, P its center; and let it be projected into *adbh*; draw the plane ABFC*a* through the center C of the ſphere, perpendicular to the plane of the given circle and plane of projection, to interſect them in the lines AB, *ab*; draw GPH, DE, perpendicular, and DQ parallel to AB; then becauſe the line GP, and the plane of projection are both perpendicular to the

the plane ABF; therefore GH is parallel to the plane of projection, and therefore to *gh*. Fig. 2.

In the circle ADB, $DQ^2 = GQH = gqh$, and $BP^2 = GP^2 = gp^2$. And (Geom. V. 12.) BP : EP or DQ : : *bp* : *ep* or *dq*, and $BP^2 : DQ^2$: : $bp^2 : dq^2$; that is, $gp^2 : gqh$: : $bp^2 : dq^2$; and therefore *agbh* is an ellipſis, whoſe tranſverſe *gh* is the diameter of the circle. *Q. E. D.*

Cor. 1. *Since ab is perpendicular to gh, therefore ab is the conjugate axis, and is twice the ſine of the* ∠ ABb *to the radius gp; that is, the conjugate axis is equal to twice the coſine of the inclination, to the radius of the circle.*

Cor. 2. *The tranſverſe axis is equal to twice the coſine of its diſtance from its parallel great circle. For* $gh = GH = 2AP =$ *twice the ſine of* AK.

Cor. 3. *The extremities of the conjugate axis are diſtant from the center of the primitive, by the ſines of the circle's neareſt and greateſt diſtance from the pole of the primitive. Thus* aC *is the ſine of* AN, *and* bC *the ſine of* BN.

Cor. 4. *Hence alſo it is plain that the conjugate axis always paſſes through the center* C *of the primitive; and is always in the line of meaſures of that circle.*

Scholium.

Every circle in the projection repreſents two equal circles, parallel and equidiſtant from the primitive. Every right line repreſents two ſemicircles, one towards the eye, the other in the oppoſite ſide. Every ellipſis repreſents two equal circles, but contrarily inclined, as AB, CD; one above the primitive, the other below it. 3.

And now the Theory being laid down, it remains only to deduce thence, ſome ſhort rules for practice, as follows.

Fig.

PROP. V. *Prob.*

5. *To project a circle parallel to the primitive.*

Rule.

Take the complement of its diſtance from the primitive, and ſet it from A to E; and with the center C and radius CD = perpendicular EF, deſcribe the circle D*g*G.

By the plane ſcale.

Take the ſine of its diſtance from the pole of the primitive; with that radius and the center C deſcribe the circle.

PROP. VI. *Prob.*

4. *To project a right circle, or one that is perpendicular to the plane of projection.*

Rule.

Through the center C of the primitive, draw the diameter AB, and take the diſtance from its parallel great circle, and ſet from A to E, and from B to D, and draw ED, the right circle required.

By the ſcale.

Take the ſine of the circle's diſtance from its parallel great circle AB, and at that diſtance draw a parallel ED for the circle required.

PROP. VII. *Prob.*

To project a given oblique circle.

Rule.

6. Draw the line of meaſures AB, and take the circle's neareſt diſtance from the primitive and ſet from

from B to D; upwards if it be above the primitive; Fig.
or downward, if below; likewiſe take its greateſt 6.
diſtance, and ſet from A to E, and draw ED, and let fall the perpendiculars EF, DG; and biſect FG in H, and erect the perpendicular KHI, making KH = HI = half ED; then deſcribe an ellipſis (by the Conic Sections) whoſe tranſverſe is IK and conjugate FG; and that ſhall repreſent the circle given.

By the ſcale.

Draw the line of meaſures AB; and take the 6.
ſines of the circle's neareſt and greateſt diſtance from the pole of the primitive, and ſet them from the center C to F and G, (both ways if the circle encompaſs the pole, but the ſame way if it lie on one ſide the pole;) biſect FG in H, and erect HK, HI perpendicular to FG, and = to the radius of the circle given, or the ſine of its diſtance from its own pole; about the axes FG, KI deſcribe an ellipſis, and it is done.

SCHOLIUM.

An ellipſis great or ſmall may be deſcribed by 10.
points, thus; through the center D of the circle and ellipſis, draw BD ⊥ the tranſverſe AR; and on AR erect a ſufficient number of perpendiculars IK, *ik* &c. and make as DB or DA : DE : : IK : IF : : *ik* : *if* &c. then through all the points E, F, *f*, &c. draw a curve. See Prop. 76. ellipſis.

PROP. VIII. *Prob.*

To find the pole of a given ellipſis.

Rule.

Through the center of the primitive C, draw the 7.
conjugate of the ellipſis; on the extreme points F, G, erect the perpendiculars FE, GD, or ſet the

Fig. 7. tranſverſe IK from E to D, and biſect ED in R, and let fall RP perpendicular to AB, then is P the pole.

By the ſcale.

7. Take CF, and CG, and apply to the ſines, and find the degrees anſwering, or the ſupplements; then take the ſine of half the ſum of theſe degrees, if F, G be both on one ſide of C, or the ſine of half the difference, if they lie on contrary ſides; and ſet it from C to the pole P.

Or thus; apply the ſemi-tranſverſe IH to the ſines, and ſet the degrees from E to R; and draw RP ⊥ to AB; and P is the pole.

PROP. IX. *Prob.*

To meaſure an arch of a parallel circle; or to ſet any number of degrees on it.

Rule.

5. With the radius of the parallel, and one foot in C deſcribe a circle Gg, draw CGB, and C*gb*; then B*b* will meaſure the given arch G*g*; or Gg will contain the given number of degrees ſet from B to *b*. So that either being given finds the other.

PROP. X. *Prob.*

To meaſure any part of a right circle.

Rule.

8. In the right circle ED, let EA = AD; and let AB be to be meaſured. Make CF = AE; with extent BA = FG deſcribe the arch GI; draw CGK to touch it in G; then is HK the meaſure of AB. For FG = S. ∠ HCK to the radius CF or AE, and BA is the ſame, by Cor. Prop. III.

Otherwiſe

Fig. 8.

Otherwiſe thus.

On the diameter ED, deſcribe the ſemicircle END, draw AN, BO, LP perpendicular to ED, then ON is the meaſure of BA, and NP of AL; and ON or NP may be meaſured as in Prop. IX.

By the ſcale.

Let AL be to be meaſured. Draw CD: and LM parallel to AC, then CM applied to the ſines gives the degrees. For radius CD : AD : : CM : AL.

Cor. *If the right circle paſſes through the center, there is no more to do, but to raiſe perpendiculars on it, which will cut the primitive as required. Or apply the part of the right circle to the line of ſines.*

PROP. XI. *Prob.*

To ſet off any number of degrees upon a right circle, DE.

Rule.

Draw CA ⊥ DE, and make the ∠ HCK = the degrees given, make CF = radius AE, take FG the neareſt diſtance, and ſet from A to B; then AB = ∠HCK, the degrees propoſed. 8.

Otherwiſe thus.

On ED deſcribe the ſemicircle END; then by Prop. IX. ſet off NP = degrees given, draw PL perpendicular to ED; then AL contains the degrees required.

Or thus by the ſcale.

Draw CD, take the given degrees off the ſines, and ſet from C to M, and draw ML parallel to CA, then AL = arch required.

Fig.

PROP. XII. *Prob.*

9. 10. *To meaſure an arch of an ellipſis; or to ſet any number of degrees upon it.*

Rule.

About AR the tranſverſe axis of the ellipſis, deſcribe a circle ABR; erect the perpendiculars BED, KFI, on AR; then BK is the meaſure of EF, or EF is the repreſentation of the arch BK. And BK is to be meaſured, or any degrees ſet upon it, as in Prop. IX.

SCHOLIUM.

Theſe Problems are all evident from the three firſt propoſitions, and need no other demonſtration. If the ſphere be projected on any plane parallel to the primitive, the projection will be the very ſame; for being effected by parallel lines, which are always at the ſame diſtance, there will be produced the ſame figure, or repreſentation. Of all orthographic projections, thoſe on the meridian, or on the ſolſtitial colure, commonly called the Analemma, is moſt uſeful; becauſe a great many of the circles of the ſphere fall into right lines or circles, whereas in the projections upon other planes, they are projected into ellipſes, which are hard to deſcribe; which makes theſe ſorts of projection to be neglected

And by the ſame rules that the circles of the ſphere are projected upon a plane, any other figure may likewiſe be orthographically projected; by letting fall perpendiculars upon the plane from all the angles, or all the points of the figure, and joining theſe points with right or curve lines, as they are in the figure itſelf.

By this kind of projection, either the convex or concave ſide of the ſphere, may be projected; which

which is peculiar to this ſort of projection; that is, either the hemiſphere towards you, or that from you, may be projected upon the plane of its great circle. And ſince in ſome caſes they both have the ſame appearance, it ought to be mentioned whether it is. Fig. 9.

But if both the convex and concave ſides of the *ſame hemiſphere* be projected; that is, if you make two projections, one for the convex, the other for the concave ſide; the circles in one will be inverted in reſpect of the other, the right to the left, &c. Becauſe in looking at the ſame hemiſphere, it will not have the ſame appearance, when you look at the contrary ſides of it; becauſe you look contrary ways at it, to ſee the external and internal ſurfaces.

SECT.

Fig.

SECT. II.

The Stereographic Projection of the SPHERE.

PROP. I.

ANY circle paſſing through the projecting point, is projected into a right line.

For all lines drawn from the projecting point, to this circle, paſs through the interſection of this circle and plane of projection, which is a right line.

Cor. 1. *A great circle paſſing through the poles of the primitive is projected into a right line paſſing through the center.*

Cor. 2. *Any circle paſſing through the projecting point is projected into a right line perpendicular to the line of meaſures, and diſtant from the center, the ſemitangent of its neareſt diſtance from the pole oppoſite*
12. *to the projecting point. Thus the circle* AE *is projected into a right line paſſing through* G, *and perpendicular to* BC, *the line of meaſures, and* GC *is the ſemitangent of* EM.

PROP. II.

Every circle (that paſſes not through the projecting point) is projected into a circle.

11. *Caſe* I. Let the circle EF be parallel to the primitive BD; lines drawn to all points of it from the projecting point A, will form a conic ſurface, which being cut parallel to the baſe by the plane BD, the ſection GH (into which EF is projected) will be a circle by the conic ſections.

Caſe

Caſe II. Let BH be the line of meaſures to the circle EF, draw FK parallel to BD, then arch AK = AF, and therefore ∠AFK or AHG = AEF; therefore in the triangles AEF, AGH, the angles at E and H are equal, and the angle A common; therefore the angles at F and G are equal. Therefore the cone of rays AEF (whoſe baſe EF is a circle) is cut by ſubcontrary ſection, by the plane of projection BD, and therefore, by the conic ſections, the ſection GH (which is the projection of the circle EF) will alſo be a circle. *Q. E. D.* Fig. 12.

Cor. *When* AF *is equal to* AG, *the circle* EF *is projected into a circle equal to itſelf.*

For then the ſimilar triangles AHG and AEF, will alſo be equal, and GH = EF.

PROP. III.

Any point on the ſphere's ſurface is projected into a point, diſtant from the center, the ſemi-tangent of its diſtance from the pole oppoſite to the projecting point.

Thus the point E is projected into G, and F into H; and CG is the ſemi-tangent of EM, and CH of MF. 12.

Cor. 1. *A great circle perpendicular to the primitive, is projected into a line of ſemi-tangents paſſing through the center, and produced infinitly.*

For MF is projected into CH its ſemi-tangent, and EM into the ſemi-tangent CG.

Cor. 2. *Any arch* EM *of a great circle perp. to the primitive, is projected into the ſemi-tangent of it.*

Thus EM is projected into GC.

Cor. 3. *Any arch* EMF *of a great circle, is projected into the ſum of its ſemi-tangents of its greateſt*

and

Fig. 12. *and leaſt diſtances from the oppoſite pole* M, *if it lie on both ſides of* M, *or the dif. of the ſemi-tangents, when all on one ſide.*

PROP. IV.

13. *The angle made by two circles on the ſurface of the ſphere, is equal to that made by their repreſentatives upon the plane of projection.*

Let the angle BPK be projected. Through the angular point P and the center C, draw the plane of a great circle PED perpendicular to the plane of projection EFG. Let a plane PHG touch the ſphere in P; then ſince the circle EPD is perpendicular both to this plane and to the plane of projection, therefore it is perpendicular to their interſection GH. The angles made by circles are the ſame as thoſe made by their tangents, therefore in the plane PGH, draw the tangents PH, PF, PG to the arches, PB, PD, PK; and theſe will be projected into the lines *p*H, *p*F, *p*K: Now I ſay the ∠ HPG = ∠ H*p*G. For the angle CPF = a right angle = C*p*A + CA*p*; therefore taking away the equal angles CPA and CAP, and ∠ *p*PF = C*p*A or P*p*F; conſequently *p*F = PF. Therefore in the right angled triangles PFG and *p*FG, there are two ſides equal and the included ∠right; therefore hypothenuſe PG = *p*G. And for the ſame reaſon in the right angled triangles PFH and *p*FH, PH = *p*H. Laſtly in the triangles PHG and *p*HG, all the ſides are reſpectively equal, and therefore ∠P = ∠*p*. *Q. E. D.*

Cor. 1. *The rumb lines projected make the ſame angles with the meridians as upon the globe; and therefore are logarithmic ſpirals on the plane of the equinoctial.*

For every particle of the rumb coincides with ſome great circle.

Cor.

Cor. 2. *The angle made by two circles on the* Fig.
ſphere, is equal to the angle made by the radii of their projections at the point of interſection.

For the angle made by two circles on a plane, is the ſame with that made by their radii drawn to the point of interſection.

PROP. V.

The center of a projected (leſſer) circle perpendicular to the primitive, is in the line of meaſures diſtant from the center of the primitive, the ſecant of the leſſer circles diſtance from its own pole; and its radius is the tangent of that diſtance.

Let A be the projecting point, EF the circle to 14.
be projected, GH the projected diameter. From the centers C, D draw CF, DF, and the triangles CFI, DFI are right angled at I; then ∠IFC = ∠FCA = 2 ∠FEA or 2FEG = 2 ∠FHG = ∠FDG, therefore IFC + IFD = FDG + IFD = a right angle; that is CFD is a right angle, and the line CD is the ſecant of BF, and the radius FD is the tangent of it. *Q. E. D.*

Cor. *If theſe circles be actually deſcribed, it is plain the radius* FD *is a tangent to the primitive at* F, *where the leſſer circle cuts it.*

PROP. VI.

The center of Projection of a great circle is in the line of meaſures, diſtant from the center of the primitive, the tangent of its inclination to the primitive; and its radius is the ſecant of its inclination.

Let A be the projecting point, EF the great 15.
circle, GH the projected diameter, D the center; draw

Fig. 15. draw DA. The angle EAF being in a ſemicircle is right. In the right angled triangle GAH, AC is perpendicular to GH, therefore ∠ GAC = AHC and their double, ECB = ADC, and their complements ECI = CAD. Therefore CD is the tangent of ECI, and radius AD its ſecant. *Q. E. D.*

16. Cor. 1. *If the great oblique circle* AGBH *be actually deſcribed upon the primitive* AIB. *I ſay, all great circles paſsing through* G *will have the centers of their projections in the line* RS *drawn through the center* D, *perpendicular to the line of meaſures* IH.

For ſince all great circles cut one another at a ſemicircle's diſtance, all circles paſſing through G muſt cut at the oppoſite point H; and therefore their centers muſt be in the line RDS.

Cor. 2. *Hence alſo if any oblique circle* GLH *be required to make any given angle with another circle* BGAH, *it will be projected the ſame way with regard to* GAH *conſidered as a primitive, and* RS *its line of meaſures; as the circle* BGA *is on the primitive* BIA, *and line of meaſures* ID. *And therefore the tangent of the angle* AGL *to the radius* GD, *ſet from* D *to* N *gives the center of* GL.

For the ∠ NGD will then be equal to AGL, by Cor. 2. Prop. IV. and therefore GLH is rightly projected.

Cor. 3. *And for the ſame reaſon, if* N *be the center of the circle* GgHR; *the centers of all circles paſſing through g and* R, *will be in the line* r N s *perpendicular to* RS; *ſo n is the center of* grR. *But then as g,* R *do not repreſent oppoſite points of the circle* GgH, *therefore all circles paſſing through g,* R, (*as* grR) *will be leſſer circles, except* GgHR.

SCHOLIUM.

Fig. 16.

Scholium.

Of all great circles in the projection, the primitive is the leaſt. For the radius of any oblique great circle (being the ſecant of the inclination) is greater than the radius of the primitive; as the ſecant is always greater than the radius. Therefore every oblique great circle in the projection is greater than the primitive.

PROP. VII.

The projected extremities of the diameter of any circle, are in the line of meaſures, diſtant from the center of the primitive circle, the ſemi-tangents of its neareſt and greateſt diſtances from the pole of projection oppoſite to the projecting point.

For the diameter of the circle EF is projected into GH, from the projecting point A. But GC is the ſemi-tangent of EB, and CH the ſemi-tangent of BF. *Q. E. D.* 15. 17.

Cor. 1. *The points where an inclined great circle cuts the line of meaſures, within and without the primitive, is diſtant from the center of the primitive, the tangent and co-tangent of half the complement of the circle's inclination to the primitive.* 15.

For CG = tangent of half EB, or of half the complement of IE the inclination. And (becauſe the ∠EAF is right) CH is the co-tangent of GAC or half EB.

Cor 2. *Hence the center* D *of a projected circle is in the line of meaſures, diſtant from the center of the primitive, half the difference of the ſemi-tangents of its neareſt and greateſt diſtance from the oppoſite pole, if it encompaſſes that pole; but half the ſum of the ſemi-tangents if it lye on one ſide the pole of projection.* 17. 18.

Cor.

Fig. Cor. 3. *And the radius is half the ſum of the ſemi-tangents, if the circle encompaſſes the pole; or half the difference if it lyes on one ſide.*

17. Cor. 4. *Hence alſo if p, q, be the projected poles, it will be* qG : pG : : qH : pH.

For draw G*n* parallel to q**A**, and ſince P, Q are the poles, therefore q**A***p* is a right angle, and ſince the angles G**A***p* and *p***A**H are equal, and G*n* perpendicular to **A***p*, therefore G**A** = **A***n*; whence by ſimilar triangles qG : qH : : **A***n* or AG : AH :: G*p*, *p*H (Geom. II. 25.) And conſequently the line qH is cut harmonically in the points G, *p*.

PROP. VIII.

The projected poles of any circle are in the line of meaſures, within and without the primitive, and diſtant from its center the tangent and co-tangent of half its inclination to the primitive.

19. The poles P, *p* of the circle EF are projected into D and *d*; and CD is the tangent of CAD or half BCP, that is, of half GCI, the inclination of the circle ICK, parallel to EF. Likewiſe C*d* is the tangent of C**A***d*, or the co-tangent of CAD. *Q. E. D.*

Cor. 1. *The pole of the primitive is its center; and the pole of a right circle is in the primitive.*

19. Cor. 2. *The projected center of any circle is always between the projected pole (neareſt to it on the ſphere) and the center of the primitive; and the projected centers of all circles lye between the projected poles.*

For the middle point of EF or its center is projected into S; and all the points in P*p* (in which are all the centers) are projected into D*d*.

Cor.

Cor. 3. *If* P *be the projected center of any circle* EFG, *any right lines* EG, FH *passing through* P *will intercept equal arches* EF, GH. Fig. 20.

For in any circle of the sphere, any two lines, passing through the center, intercept equal arches; and these are projected into right lines, passing through the projected center P, and therefore EF, GH, represent equal arches.

PROP. IX.

If EFGH, *efgh represent two equal circles, whereof* EFG *is as far distant from its pole* P, *as efg is from the projecting point. I say, any two right lines (*eEP, *and* fFP,*) being drawn through* P, *will intercept equal arches (in representation) of these circles; on the same side, if* P *falls within the circles; but on the contrary side, if without; that is,* EF = *ef, and* GH = *gh.* 20. 21.

For by the nature of the section of a sphere; any two circles passing through two given points or poles on the surface of the sphere, will intercept equal arches of two other circles equidistant from these poles. Therefore the circles EFG and *efg* on the sphere, are equally cut by the planes of any two circles passing through the projecting point and the pole P, on the sphere. But these circles (by Prop. I.) are projected into the right lines P*e* and P*f*, passing through *p*. And the intercepted arches representing equal arches on the sphere, are therefore equal, that is, EF = *ef*, and GH = *gh*.

Cor. 1. *If a circle is projected into a right line* EF, *perpendicular to the line of measures* EG; *and if from the center* C *a circle ef* P *be described passing through its pole* P, *and* P*f be drawn; then arch ef* = EF. *And if any other circle be described whose vertex is* P, *the arch ef will always be equal to* EF. 22.

Fig. Cor. 2. *Hence alſo, if from the pole of a great circle there be drawn two right lines, the intercepted arch of the projected great circle will be equal to the intercepted arch of the primitive.*

23. Cor. 3. *After the ſame manner if there be two*
24. *equal circles* EF, *ef, whereof one is as far from the pole* P, *as the other is from the pole of projection* C, *oppoſite to the projecting point. Then any circle drawn through the points* P, C, *will intercept equal arches* EF = *ef; and* GH = *gh, between it and the line of meaſures* PCG.

For this is true on the ſphere, and their projections are the ſame.

Cor. 4. *If from an angular point be drawn two right lines through the poles of its ſides; the intercepted arch of the primitive, will be equal to that angle.*

For the diſtance of the poles is equal to that angle.

PROP. X.

25. *If* QH, NK *be two equal circles, whereof* NK
26. *is as far from the projecting point as* QH *from its pole* P; *and if they be projected into the circles whoſe radii are* MC *or* CL, *and* DF *or* FG, F *being the center of* DG, *and* E *the projected pole. I ſay, the pole* E *will be diſtant from their centers in proportion to the radii of the circles; that is,* CE : EF : : CL : DF *or* FG.

For ſince NK and ML are parallel, and arch NI = PH, therefore ∠ELI = NKI (or *n*KI) = GIP; therefore the triangles IEL and IEG are ſimilar, whence EL : EI : : EI : EG. Again the angle EMI = KNI = PIQ, and therefore the triangles IEM and IED are ſimilar, whence EM : EI : : EI : ED. Therefore $EI^2 = EL \times EG =$ EM

EM × ED. Consequently EM : EL : : EG : ED; Fig.
and by composition $\frac{EM+EL}{2} : \frac{EM-EL}{2} : :$
$\frac{EG+ED}{2} : \frac{EG-ED}{2}$; that is, CM : EC : : FG
: EF. *Q. E. D.*

Cor. 1. *Hence if the circle* KN *be as far from* 25.
the projecting point, as QH *is from either of its poles,* 26.
and if E, O, *be its projected poles; then will* EL :
EM : : ED : : EG : : OD : OG.

This follows from the foregoing demonstration, and Cor. 4. Prop. VII.

Cor. 2. *Hence also if* F *be the center, and* FD *the* 25.
radius of any circle QH, *and* E, O *the projected* 26.
poles; then EF : DF : : DF : FO.

For it follows from Cor. 1, that $\frac{EG+ED}{2}$:
$\frac{EG-ED}{2} : : \frac{OG+OD}{2} : \frac{OG-OD}{2}$.

Cor. 3. *Hence if the circle* DBG *be as far from* 27.
its projected pole P, *as* LMN *is from the projecting* 28.
point; and if any right lines be drawn through P, *as* MPG, NPK, *they will cut off similar arches* GK, MN *in the two circles.*

For from the centers C, F, draw the lines CN, FK, then since the angles CPN, and FPK are equal, and by this Prop. CP : CN : : FP : FK; therefore (Geom. II. 16.); the triangles PCN and PFK are similar; and the angle PCN = ∠PFK; therefore the arches MN and GK are similar.

Cor. 4. *Hence also if through the projected pole* P 27.
of any circle DBG, *a right line* BPK *be drawn.* 28.
Then I say the degrees in the arch GK *shall be the measure of* DB *in the projection. And the degrees in* DB, *shall be the measure of* GK *in the projection.*

Fig. For (by Prop. IX.) the arch MN is the meaſure of DB, and therefore GK which is ſimilar to MN will alſo be the meaſure of it.

Cor. 5. *The centers of all projected circles are all beyond the projected poles (in reſpect to the center of the primitives); and none of their centers can fall between them.*

20. Cor. 6. *Hence it follows (by* Cor. 5. *and* Pr. VIII. Cor. 3.*) that all circles that are not parallel to the primitive have equal arches on the ſphere repreſented by unequal arches on the plane of projection.*

For if P be the projected center, then GH is greater than EF.

Scholium.

It will be eaſy by the foregoing propoſitions to deſcribe the repreſentation of any circle, and the reverſe will eaſily ſhow what circle of the ſphere any projected circle repreſents. What follows hereafter is deduced from the foregoing propoſitions and will eaſily be underſtood without any other demonſtration.

If the ſphere was to be projected on any plane parallel to the primitive, it is all the ſame thing. For the cones of rays iſſuing from the projecting point, are all cut by parallel planes into ſimilar ſections, it only makes the projections bigger or leſs, according to the diſtance of the plane of projection, whilſt they are ſtill ſimilar; and amounts to no more than projecting from different ſcales upon the ſame plane. And therefore the projecting the ſphere on the plane of a leſſer circle is only projecting it upon the great circle parallel thereto, and continuing all the lines of the ſcheme to that leſſer circle.

PROP.

Fig.

PROP. XI. *Prob.*

To draw a circle parallel to the primitive at a given diſtance from its pole.

Rule.

Through the center C draw two diameters AB, DE, perpendicular to one another. Take in your compaſſes the diſtance of the circle from the pole of the primitive oppoſite to the projecting point, and ſet it from D to F; from E draw EF to interſect AB in I; with the radius CI, and center C, deſcribe the circle GI required. 29.

By the plane Scale.

With the radius CI, equal to the ſemi-tangent of the circle's diſtance from the pole of projection oppoſite the projecting point, deſcribe the circle IG. Here the radius of projection CA, is the tangent of 45°, or the ſemi-tangent of 90°.

PROP. XII. *Prob.*

To draw a leſſer circle perpendicular to the primitive at a given diſtance from the pole of that circle.

Rule.

Through the pole B draw the line of meaſures AB, make BG the circle's diſtance from its pole, and draw CG, and GF perpendicular to it; with the radius FG deſcribe the circle GI required. 30.

By the Scale.

Set the ſecant of the circle's diſtance from its pole from C to F, gives the center. With the tangent of that diſtance for a radius, deſcribe the circle GI.

Or thus, make BG the circle's diſtance from its pole; and GF its tangent, ſet from G, gives F the

Fig. 30. center; through G deſcribe the circle GI from the center F.

Cor. *Hence a great circle perpendicular to the primitive, is a right line* CDE *drawn through the center perpendicular to the line of meaſures.*

Scholium.

When the center F lyes at too a great a diſtance; draw EG, to cut AB in H; or lay the ſemi-tangent of DG from C to H. And through the three points G, H, I, draw a circle with a bow.

PROP. XIII. *Prob.*

To deſcribe an oblique circle at a given diſtance from a pole given.

Rule.

31. Draw the line of meaſures AB through the given point *p*, if that point is given; and draw DE ⊥ to it, alſo draw E*p*P. Or if the point *p* is not given, ſet the height of the pole above the primitive from B to P. Then from P ſet of PH = PI = circle's diſtance from its pole; and draw EH, EI, to interſect AB in F and G. About the diameter FG deſcribe the circle required.

By the Scale.

If the point P is given, apply C*p* to the ſemi-tangents, and it gives the diſtance of the pole from D, the pole of projection oppoſite to the projecting point. This diſtance being had, you will eaſily find the greateſt and neareſt diſtances of the circle from the pole of the primitive oppoſite to the projecting point; take the ſemi-tangents of theſe diſtances and ſet from C to G and F, both the ſame way if the circle lye all on one ſide, but each its own way, if on different ſides of D. And then FG is the diameter of the circle required to be drawn.

Cor.

Cor. 1. *If* F *be the pole of a great circle, as of* DLE. *Draw* EFH, *and make* HP = DH, *and draw* EpP, *and then P is its center.* Fig. 31.

Or thus, draw EFH *through the pole* F, *make* HK 90 *degrees; draw* EK *cutting the line of meaſures in* L. *Through the three points* D, L, E, *draw the great circle required.*

Cor. 2. *Hence it will be eaſy to draw one circle parallel to another.*

PROP. XIV. *Prob.*

Through two given points A, B *to draw a great circle.*

Rule.

Through one of the points A, draw a line through the center, ACG; and EF perpendicular to it. Then draw AE, and EG perpendicular to it. Through the three points A, B, G draw the circle required. 32.

Or thus; From E (found as before) draw EH, and then HCI, and laſtly EIG, gives G a third point, through which the circle muſt paſs.

By the ſcale.

Draw ACG; and apply AC to the ſemi-tangents, find the degrees, ſet the ſemi-tangent of its ſupplement from C to G, for a third point.

Or thus; Apply AC to the tangents, and ſet the tangent of its complement from C to G. And through the three points ABG, deſcribe the circle required.

For ſince HEI or AEG is a right angle, therefore A, G are oppoſite points of the ſphere; and therefore all circles paſſing through A and G are great circles.

Scholium.

If the points A, B, G lie nearly in a right line, then you may draw a circle through them with a bow.

Fig.

PROP. XV. *Prob.*

About a pole given, to deſcribe a circle through a given point.

Rule.

33. Let P be the pole, and B the given point; through P, B deſcribe the great circle AD (by Prop. XIV.) whoſe center is E; through the center C draw CPH; and from the center E, draw EB to the point B, and BF perpendicular to it, cutting CH in F. To the center F, and radius FB deſcribe the circle BGH required.

PROP. XVI. *Prob.*

To find the poles of any circle FNG.

Rule.

31. Through its center drawn the line of meaſures AG, and DE perpendicular to it. Draw EFH, and ſet its diſtance (from its own pole) from H to P, and draw E*p*P, then *p* is the pole.

Or thus, Draw EFH, EIG, and biſect HI in P, and draw E*p*P, and *p* is the internal pole. Laſtly draw PCQ, and EQ *q*, and *q* is the external pole.

In a great circle DLE, draw ELK, and make DH = AK, or KH = AD, and draw EFH, and F is the pole.

By the Scale.

Apply CF to the ſemi-tangents, and note the degrees. Take the ſum of theſe degrees and of the circle's diſtance from its pole, if the circle lie all on one ſide, but their difference if it encompaſſes the pole of projection; ſet the ſemi-tangent of this ſum or difference from C to the internal pole *p*. And the ſemi-tangent of its ſupplement C*q*, gives the external pole *q*.

Or

Or thus, Apply CF and CG to the ſemi-tangents, ſet the ſemi-tangent of half the ſum of the degrees (if the circle lies all one way) or of half the difference (if it encompaſſes the pole of projection), from C to the pole *p*; and the ſemi-tangent of the ſupplement, C*q* gives the external pole *q*. Fig. 31.

In a great circle as DLE, draw the line of meaſures AB perp. to DE; and ſet the tangent and co-tangent of half its inclination, from the center C, different ways to F and *f*; which gives the internal and external poles F and *f*.

PROP. XVII. *Prob.*

To draw a great circle at any given inclination above the primitive; or making any given angle with it, at a given point.

Rule.

Draw the line of meaſures AB; and DCE perpendicular to it. Make EK = 2HD = twice the complement of the circle's inclination; (or DK = 2AH = twice the inclination); and draw EKF, then F is the center of EGD, the circle required. 34.

Or thus; Draw DE and AB perp. to it, and let D be the point given. Make AH the inclination and draw EGH and HCN; and ENO, to cut AB in O. Then biſect GO in F, for the center of the circle required.

By the Scale.

Set the tangent of the inclination in the line of meaſures from C to F, then F is the center. Set the ſemi-tangent of the complement from C to G; then GF or DF is the radius.

Or the ſecant of the inclination ſet from G or D to F gives the center.

Cor.

Fig. Cor. *To draw an oblique circle to make a given angle with a given oblique circle* DGE *at* D.

Draw EGH, and ſet the given angle from H to I, and draw ELI. Through D, L, E deſcribe a great circle.

But if the circle DLE was to lie on the other ſide of DGE, the arch HI muſt be taken on the other ſide of H.

PROP. XVIII. *Prob.*

Through a given point P, *to draw a great circle, to make a given angle with the primitive.*

Rule.

35. Through the point given P, and thecenter C draw the line AB; and DE perpendicular to it. Set the given angle from A to H and from H to K, and draw BGK; with radius CG, and center C deſcribe the circle GIF; and with radius BG and center P croſs that circle in F. Then with radius FP and center F, deſcribe the circle LPM required.

By the Scale.

With the tangent of the given angle and one foot in C, deſcribe the arch FG. With the ſecant of the given angle and one foot in the given point P, croſs that arch at F. From the center F deſcribe a circle through the point P.

PROP. XIX. *Prob.*

To draw a great circle to make a given angle with a given oblique circle FPR, *at a given point* P, *in that circle.*

Rule.

36. Through the center C and the given point P, draw the right line DE; and AB perpendicular to it;

it; draw APG and make BM = 2DG; and draw AM to cut DE in I. Draw IQ perpendicular to DE, then IQ is the line wherein the centers of all circles are found which pass through the point P. Find N the center of the given circle FPR, and make the angle NPL equal to the given angle, then L is the center of the circle HPK required. Fig. 36.

By the Scale.

Through P and C draw DE; apply CP to the semi-tangents, and set the tangent of its complement from C to I (or the secant from P to I). On DI erect the perpendicular IQ. Find the center N of FPR, and make the angle NPL = angle given, and L is the center.

Cor. *If one circle is to be drawn perpendicular to another, it must be drawn through its poles.*

PROP. XX. *Prob.*

To draw a great circle through a given point P, *to make a given angle with a given circle* DE.

Rule.

About the given point P as a pole (by Prop. 13. Cor. 1.) describe the great circle FG; find I the pole of the given circle DE, and (by Prop. 16.) about the pole I (by Prop. 13.) describe the small circle HKL at a distance equal to the given angle, to intersect FG in H; about the pole H describe (by Prop. 13.) the great circle APB required. 37.

PROP. XXI. *Prob.*

To draw a great circle to cut two given great circles abd, ebf at given angles.

Rule.

Find the poles *s*, *r*, of the two given circles, by Prop. 16. about which draw two parallels *phk*, *pnk*, 50.

Fig. 50. *pnk*, at the distances respectively equal to the angles given by Prop. 13. the point of intersection *p*, is the pole of the circle *moq* required.

Cor. *Hence, to draw a right circle to make with an oblique circle, abd, any given angle. Draw a parallel phk at a distance from the pole of the oblique circle, equal to the given angle. Its intersection f with the primitive, gives the pole of the right circle gCt required.*

PROP. XXII. *Prob.*

To lay any number of degrees on a great circle, or to measure any arch of it.

Rule.

38. Let AFI be the primitive; find the internal pole P of the given circle DEH, or (by Prop. 16.) lay the degrees on the primitive from A to F, and draw PA, PF, intercepting the part required DE. Or to measure DE, draw PEF and PDA, and AF is its measure, and applied to the line of chords shows how many degrees it is.

Or thus; Find the external pole *p* of the given circle, set the given degrees from I to K, and draw *p*I, *p*K, intercepting the part DE required. Or to measure DE, through D and E draw *p*I, *p*K, then KI is the measure of DE.

Or thus; Through the internal pole P, draw the lines DPG, and EPL; setting the given degrees from G to L in the circle GL; then DE is the arch required. Or if DE be to be measured, then the degrees in the arch GL is the measure of DE.

Or thus; Set the given degrees from G to H in the circle GL and from the external pole *p*, draw *p*G, *p*H, intercepting DE the arch required. Or to measure DE, draw *p*DG, *p*EH, then the degrees in GH, is equal to DE.

For

For a right Circle. Fig.

Through C (fig. 34.) draw DE perpendicular to AB, then E is the pole of AB. Set the number of degrees from D to H, and draw EGH, then GC contains the degrees required. Or if GC be given, draw EGH, then HD in degrees, is the meaſure of GC.

And in like manner AG contains the degrees in AH, or AH is the meaſure of AG.

By the Scale for right Circles.

Let CA be the right circle, take the number of degrees off the ſemi-tangents and ſet from C to D for the arch CD. Or if the given degrees are to be ſet from A, then take the degrees off the ſemi-tangents from 90° towards the beginning, and ſet from A to D. And if CD was to be meaſured, apply it to the beginning of the ſemi-tangents; and to meaſure AD, apply it from 90° backwards, and the degrees intercepted gives its meaſure. 38.

SCHOLIUM.

The primitive is meaſured by the line of chords, or elſe it is actually divided into degrees.

PROP. XXIII. *Prob.*

To ſet any number of degrees on a leſſer circle, or to meaſure any arch of it.

Rule.

Let the leſſer circle be DEH; find its internal pole P, by Prop. 16. deſcribe the circle AFK parallel to the primitive, by Prop. 11. and as far from the projecting point, as the given circle DE is from its internal pole P, ſet the given degrees from A to F, and draw PA, PF interſecting the given circle in 38.

Fig. 38. in D, E; then DE is the arch required. Or to meaſure DE, draw PDA, PEF, and AF ſhows the degrees in DE.

Or thus; Find the external pole *p*, of the given circle by Prop. 16. deſcribe the leſſer circle AFK as far from the projecting point, as DE the given circle is from its pole *p* by Prop. 11. ſet the degrees from I to K and draw *p*DI, *p*EK, then DE repreſents the given number of degrees. Or to meaſure DE; draw *p*DI, *p*EK; and KI is the meaſure of DE.

Or thus; Let O be the center of the given circle DEH; through the internal pole P, draw lines DPG, EPL, divide the quadrant GQ into 90 equal degrees, and if the given degrees be ſet from G to L, and LPE drawn, then DE will repreſent theſe degrees. Or the degrees in GL will meaſure DE.

Or thus; Divide the quadrant GR into 90 equal parts or degrees, and ſet the given degrees from G to H, and draw *p*DG, *p*EH, from the external pole *p*; then DE will repreſent the given degrees. Or through D, E drawing *p*DG, *p*EH, then the number of equal degrees in GH is the meaſure of DE.

Scholium.

Any circle parallel to the primitive is divided or meaſured, by drawing lines from the center, to the like diviſions of the primitive. Or by help of the chords on the ſector, ſet to the radius of that circle.

PROP. XXIV. *Prob.*

To meaſure any angle.

Rule.

By Cor. 1. Prop. 13. About the angular point as a pole, deſcribe a great circle, and note where it interſects the legs of the angle; through theſe points of

of interſection, and the angular point, draw two right lines, to cut the primitive; the arch of the primitive intercepted between them is the meaſure of the angle. This needs no example. Fig.

Or thus; by Prop. 16. Find the two poles of the containing ſides, (the neareſt, if it be an acute angle, otherwiſe the furtheſt) and through the angular point and theſe poles, draw right lines to the primitive, then the intercepted arch of the primitive is the angle required. As if the angle AEL was required. Let C and F be the poles of EA and EL. From the angular point E, draw ECD and EFH. Then the arch of the primitive DH, is the meaſure of the angle AEL. 31.

Scholium.

Becauſe in the Stereographic Projection of the Sphere, all circles are projected either into circles or right lines, which are eaſily deſcribed; therefore this ſort of projection is preferred before all others. Alſo thoſe planes are preferred before others to project upon, where moſt circles are projected into right lines, they being eaſier to deſcribe and meaſure than circles are; ſuch are the projections on the planes of the meridian and ſolſtitial colure.

SECT.

Fig.

SECT. III.

The Gnomonical Projection of the SPHERE.

PROP. I.

39. *EVERY great circle as* BAD *is projected into a right line, perpendicular to the line of meaſures, and diſtant from the center, the co-tangent of its inclination, or the tangent of its neareſt diſtance from the pole of projection.*

Let CBED be perpendicular both to the given circle BAD and plane of projection, and then the interſection CF will be the line of meaſures. Now ſince the plane of the circle BD, and the plane of projection are both perpendicular to BCDE, therefore their common ſection will alſo be perpendicular to BCDE, and conſequently to the line of meaſures CF. Now ſince the projecting point **A** is in the plane of this circle, all the points of it will be projected into that ſection; that is, into a right line paſſing through *d*, and perpendicular to C*d*. And C*d* is the tangent of CD, or co-tangent of C*d*A. *Q. E. D.*

39. Cor. 1. *A great circle perpendicular to the plane of projection is projected into a right line paſſing through the center of projection; and any arch is projected into its correſpondent tangent.*

Thus the arch CD is projected into the tangent C*d*.

Cor. 2. *Any point as* D, *or the pole of any circle, is projected into a point d diſtant from the pole of projection* C, *the tangent of that diſtance.*

Cor 3. *If two great circles be perpendicular to each other, and one of them paſſes through the pole of projection;*

tion; they will be projected into two right lines perpendicular to each other. Fig. 39.

For the repreſentation of that circle which paſſes through the pole of projection is the line of meaſures of the other circle.

Cor. 4. *And hence if a great circle be perpendicular to ſeveral other great circles, and its repreſentation paſs through the center of projection; then all theſe circles will be repreſented by lines parallel to one another, and perpendicular to the line of meaſures or repreſentation of that firſt circle.*

PROP. II.

If two great circles interſect in the pole of projection; their repreſentations ſhall make an angle at the center of the plane of projection equal to the angle made by theſe circles on the ſphere. 39.

For ſince both theſe circles are perpendicular to the plane of projection; the angle made by their interſections with this plane, is the ſame as the angle made by theſe circles. *Q. E. D.*

PROP. III.

Any leſſer circle parallel to the plane of projection is projected into a circle, whoſe center is the pole of projection, and radius the tangent of the circle's diſtance from the pole of projection. 39.

Let the circle PI be parallel to the plane GF. Then the equal arches PC, CI are projected into the equal tangents GC, CH; and therefore C the point of contact and pole of the circle PI and of the projection, is the center of the repreſentation GH. *Q. E. D.*

Cor. *If a circle be parallel to the plane of projection, and* 45 *degrees from the pole, it is projected into*

Fig. *a circle equal to a great circle of the ſphere; and may therefore be looked upon as the primitive circle in this projection, and its radius the radius of projection.*

PROP. IV.

40. *Every leſſer circle (not parallel to the plane of projection) is projected into a conic ſection, whoſe tranſverſe axis is in the line of meaſures, and whoſe neareſt vertex is diſtant from the center of the plane the tangent of its neareſt diſtance from the pole of projection; and the other vertex is diſtant the tangent of its furtheſt diſtance.*

Let BE be parallel to the line of meaſures *dp*, then any circle is the baſe of a cone whoſe vertex is at A, and therefore that cone being produced will be cut by the plane of projection in ſome conic ſection; thus the circle whoſe diameter is DF will be cut by the plane in an ellipſis whoſe tranſverſe is *df*; and C*d* is the tangent of CAD, and C*f* of CF. In like manner the cone AFE being cut by the plane, *f* will be the neareſt vertex; and the other point into which E is projected, is at an infinite diſtance. Alſo the cone AFG (whoſe baſe is the circle FG) beicg cut by the plane *f* is the neareſt vertex; and GA being produced gives *d* the other vertex. *Q. E. D.*

Cor. 1. *If the diſtance of the furtheſt point of the circle be leſs than* 90° *from the pole of projection, then it will be projected into an ellipſis.*

Thus DF is projected into *df*, and DC being leſs than 90°, the ſection *df* is an ellipſis, whoſe vertices are at *d* and *f*; for the plane *df* cuts both ſides of the cone, *d*A, *f*A.

Cor. 2. *If the furtheſt point be more than* 90 *degrees from the pole of projection, it will be projected into*

into an hyperbola. Thus the circle FG *is projected into an hyperbola whoſe vertices are f and d, and tranſverſe fd.* Fig. 40.

For the plane *dp* cuts only the ſide A*f* of the cone.

Cor. 3. *And in the circle* EF, *where the furtheſt point* E *is* 90° *from* C; *it will be projected into a parabola, whoſe vertex is f.*

For the plane *dp* (cutting the cone FAE) is parallel to the ſide AE.

Cor. 4. *If* H *be the center, and* K, *k*, *l*, *the focus of the ellipſis, hyperbola, or parabola; then* $HK = \frac{Ad - Af}{2}$ *for the ellipſis, and* $Hk = \frac{Ad + Af}{2}$ *for the hyperbola; and (drawing fn perpendicular on* AE) $fl = \frac{nE + Ff}{2}$, *for the parabola; which are the repreſentations of the circles* DF, FG, FE *reſpectively.*

This all appears from the Conic Sections.

PROP. V.

Let the plane TW *be perpendicular to the plane of projection* TV, *and* BCD *a great circle of the ſphere in the plane* TW. *And let the great circle* BED *be projected in the right line bek. Draw* CQ*s* ⊥ *bk, and* C*m* ‖ *to it and equal to* CA; *and make* Q*s* = Q*m*; *then I ſay any angle* Q*st* = Q*t*. 41.

Suppoſe the hypothenuſe AQ to be drawn, then ſince the plane ACQ is perpendicular to the plane TV, and *b*Q is ⊥ to the interſection CQ, therefore *b*Q is perpendiclar to the plane ACQ, and conſequently *b*Q is perpendicular to the hypothenuſe AQ. But AQ=Q*m*=Q*s*, and Q*s* is alſo perpendicular to *b*Q. Therefore all angles made at S, cut the line *b*Q in the ſame points as the angles made

Fig. 41. at A; but by the angles at A the circle BED is projected into the line *b*Q. Therefore the angles at *s* are the measures of the parts of the projected circle *b*Q; and *s* is the dividing center thereof. *Q. E. D.*

Cor. 1. *Any great circle tQb is projected into a line of tangents to the radius* SQ.

For Q*t* is the tangent of the angle QS*t* to the radius QS or Q*m*.

Cor. 2. *If the circle b*C *pass through the center of projection; then* A *the projecting point is the dividing center thereof. And* C*b is the tangent of its correspondent arch* CB, *to* CA *the radius of projection.*

53-2. Cor. 3. *If* P *be the point where the great circles* PC, PF, *&c. meet (as at the pole); and if* CB, *the radius of projection, be perpendicular to* CP, *at* C *the point of contact. Then the dividing center* A *of* PF, *is distant from* P, *the length of* PB.

For ICA is perpendicular to PF, and IA = FC; whence $PA^2 = PI^2 + IA^2 = PI^2 + FC^2 = PI^2 + FI^2 + IC^2 = PC^2 + FI^2 = PC^2 + CB^2 = PB^2$, and PA = PB.

PROP. VI.

41. *Let the parallel circle* GEH *be as far from the pole of projection* C *as the circle* FKI *is from its pole* P; *and let the distance of the poles* C, P *be bisected by the radius* AO, *and draw b*AD *perpendicular to* AO; *then any right line bek drawn through b, will cut off the arches hl* = F*n, and ge* = *kf (supposing f the other vertex), in the representations of these equal circles in the plane of projection.*

For let G, E, R, L, H, N, R, K, I be respectively projected into the points *g*, *e*, *r*, *l*, *h*, *n*, *r*, *k*, *f*. Then since in the sphere, the arch BF = DH, and arch BG = DI. And the great circle BEKD makes the angles at B and D equal, and

is

is projected into a right line as *bl*; therefore the triangular figures BFN and DHL are ſimilar, and equal; and likewiſe BGE and DIK are ſimilar and equal, and LH = NF, and KI = EG; whence it is evident their projections *lh* = *n*F, and *kf* = *ge*. *Q. E. D.* Fig. 41.

PROP. VII.

If hlg and F*nk be the projections of two equal circles, whereof one is as far from its pole* P *as the other from its pole* C; *which is the center of projection*; *and if the diſtance of the projected poles* C, *p be divided in o, ſo that the degrees in* C*o*, *op*, *be equal, and the perpendicular o*S *be erected to the line of meaſures gh. I ſay the lines pn,* C*l, drawn from the poles* C, *p through any point* Q *in the line o*S, *will cut off the arch* F*n* = ∠Q*C*p. 42.

For drawing the great circle GPI, in a plane perpendicular to the plane of projection. The great circle AO perpendicular to CP is projected into *o*S by Prop. I. Cor. 3. Now let Q be the projection of *q*, and ſince *p*Q, CQ are right lines, therefore they repreſent the great circles P*q*, C*q*. But the ſpherical triangle P*q*C is an iſoceles-triangle, and therefore the angles at P and C are equal. But becauſe P is the pole of FI, therefore the great circle P*q* continued, will cut an arch off FI = ∠CP*q* = ∠PC*q* = ∠Q*C*p by Prop. II. That is (ſince F*n* repreſents the part cut off from FI) arch F*n* = arch *lh* or ∠Q*C*h. *Q. E. D.*

Cor. *Hence if from the projected pole p of any circle, a perpendicular be erected to the line of meaſures*; *it will cut off a quadrant from the repreſentation of that circle.*

For that perpendicular will be parallel to *o*S; Q being at an infinite diſtance.

Fig.

PROP. VIII.

42. *Let* F*nk be the projection of any circle* FI, *and p the projected pole* P. *And if* C*g be the co-tangent of* CAP, *and g*B *perpendicular to the line of meaſures g*C, *and* CAP *be biſected by* AO, *and the line o*B, *be drawn to any point* B, *and alſo p*B *cutting* F*nk in d. I ſay the angle go*B = *arch* F*d.*

For the arch PG is a quadrant, and the $\angle goA = \angle gpA + \angle oAp =$ (becauſe GCA and *g*A*p* are right angles) $gAC + oAp = gAC + CAo = \angle gAo$. Therefore $gA = go$, conſequently *o* is the dividing center of *g*B the repreſentation of GA; and conſequently by Prop. V. $\angle goB$ is the meaſure of *g*B. But ſince *pq* repreſents a quadrant, therefore *p* is the pole of *g*B, and therefore the great circle *pd*B paſſing through the pole of the circles *g*B and F*n* will cut off equal arches in both, that is $Fd = gB = \angle goB$. *Q. E. D.*

Cor. *The* $\angle goB$ *is the meaſure of the angle gp*B.

For the triangle *gp*B repreſents a triangle on the ſphere wherein the arch which *g*B repreſents is equal to the angle which $\angle p$ repreſents, becauſe *gp* is 90 degrees. Therefore *go*B is the meaſure of both.

SCHOLIUM.

Thus far I have treated of the theory; what follows is the practical part, and depends altogether on what is above delivered, in which I think no difficulty can occur. In the Gnomonical Projection, the plane projected on, is ſuppoſed to touch the hemiſphere to be projected, in its vertex; and the point of contact will be *the center of projection.* But if it be required to project upon any plane parallel

rallel to this touching plane, the proceſs will be no way different, and is only taking a greater or leſſer radius of projection, according to the greater or leſſer diſtance; which is in effect projecting a greater or leſſer ſphere upon its touching plane. Fig. 42.

When you have the ſphere to project this way, upon a given plane; it will aſſiſt the imagination, if you ſuppoſe yourſelf placed in the center of the ſphere with your face towards the plane, whoſe poſition is given; and from thence projecting with your eye, the circles of the ſphere upon this plane.

PROP. IX. *Prob.*

To draw a great circle, through a given point, and at a given diſtance from the pole of projection. 43.

Rule.

Deſcribe the circle ADB with the radius of projection, and through the given point P draw the right line PCA, and CE perpendicular to it; make the angle CAE=given diſtance of the circle from C, and through E deſcribe the circle EFG and through P draw the line PK touching the circle in I, then is PIK the circle required.

By the plane Scale.

With the tangent of the circle's diſtance from the pole of projection C, deſcribe the circle EIF, and draw PK to touch this circle; and PIK is the circle required.

PROP. X. *Prob.*

To draw a great circle perpendicular to a given great circle, which paſſes through the pole of projection; and at a given diſtance from that pole. 43.

Rule.

Draw the primitive ADB. Let CI be the given circle, draw CL perpendicular to CI, and make the

Fig. 43. angle CLI = the given distance; through I draw KP parallel to CL for the circle required.

By the Scale.

In the given circle CI, set the tangent of the given distance, from C to I; through I draw KP perpendicular to CI, then KP is the circle required.

PROP. XI. *Prob.*

44. *To measure any part of a great circle; or to set any number of degrees thereon.*

Rule.

Let EP be the great circle; through C draw ID perpendicular to EP, and CB parallel to it. Let EBD be a circle described with the radius of projection CB, make IA=IB; then A is the dividing center of EP, consequently drawing AP, the ∠IAP=measure of the given arch IP.

Or thus, Set CD from I to F, and FC from I to A, and draw AP, &c.

Or if the degrees be given, make the ∠IAP = the given degrees, which cuts off IP, the arch correspondent thereto.

By the Scale.

Draw ICD perpendicular to EP; apply CI to the tangents, and set the semi-tangent of its complement from C to A, gives the dividing center of EP, &c.

PROP. XII. *Prob.*

51. *To draw a great circle to make a given angle with a given great circle, at a given point; or to measure an angle made by two great circles.*

Rule.

Let P be the given point, and PB the given great circle. Draw through P, and C the center of projection,

jection, the line PCG, to which from C draw CA perpendicular, and equal to the radius of projection. Draw PA and AG perpendicular to it, at G erect BD perpendicular to GC, cutting PB in B; draw AO bisecting the angle CAP; then at the point O, make BOD = angle given, and from D draw the line DP, then BPD is the angle required. Fig. 51.

Or if the degrees in the angle BPD be required, from the points B, D, draw the lines BO, DO; and the angle BOD is the measure of BPD.

Cor. *If an angle be required to be made at the pole or center of projection, equal to a given angle; this is no more than drawing two lines from the center making the angle required. And if one great circle be to be drawn ⊥ to another great circle, it must be drawn through its pole.*

PROP. XIII. *Prob.*

To project a lesser circle parallel to the primitive. 43.

Rule.

With the radius of projection AC, and center C, describe the primitive circle ADB, by Cor. Prop. III. and draw ACB, and GCE perpendicular to it.

Set the circle's distance from its pole from B to H, and from H to D, and draw AFD. With radius CE describe the circle EFG required.

By the Scale.

With the radius CE equal to the tangent of the circle's distance from its pole, describe the circle EFG, for the circle required.

PROP. XIV. *Prob.*

To draw a lesser circle perpendicular to the plane of projection. 48.

Rule.

Through the center of projection C, draw its parallel great circle TI. At C make the angle ICN

and

Fig. 48. and TCO = the given circle's diſtance from its parallel great circle TI; make CL equal radius of projection, and draw LM perpendicular to CL. Set LM from C to V, or CM from C to F. Then through the vertex V between the aſymptotes CN, CO deſcribe the hyperbola WVK. Or to the focus F, and ſemi-tranſverſe CV, deſcribe the hyperbola; for the circle required.

Otherwiſe by Points.

Through the center of projection C draw the line of meaſures FC, and TCI perpendicular to it, draw any number of right lines CV, DE, GH, IK &c. and PQ, RS, TW, &c. perpendicular to TI. And by Prop. XI. make CV, DE, GH, &c. each equal to the diſtance of the given circle from its paralle! great circle; then all the points W, S, Q, V, E, H, K, &c. joined by a regular curve will be the repreſentation of the circle required.

Or thus.

Make the angle *iak* = diſtance of the given circle from its parallel great circle. Then through the center of projection C, draw the great circle TCI parallel to the circle given, upon which erect the perpendicular CA = radius of projection. Alſo draw any number of right lines CV, DE, GH, IK, &c. perpendicular to TI. Then take each of the diſtances from A to C, D, G, I, &c. and ſet them from *a* to *c*, *g*, *d*, *i*, &c. and to *ai* draw the perpendiculars *cv*, *dc*, *gh*, *ik*, &c. and make CV, DE, GH, IK, &c. reſpectively equal to *cv*, *de*, *gh*, *ik*, &c. which gives the points V, E, H, K, &c. after the ſame manner on the other ſide, find the points Q, S, W, &c. then through all theſe points W, S, Q, V, E, H, K, &c. draw a regular curve, which will be an hyperbola repreſenting the circle given.

By

By the Scale. Fig. 48.

Take the tangent of the circle's diſtance from its parallel great circle, and ſet it from C (the center of projection) to V, and the ſecant thereof from C to F. Then with the ſemi-tranſverſe CV, and focus F, deſcribe the hyperbola WVHK.

PROP. XV. *Prob.*

To project any leſſer oblique circle given.

Rule.

Draw the line of meaſures *dp*, and at C the center of projection draw CA ⊥ to *dp* and = radius of projection; with the center A, deſcribe the circle DCFG; and draw RAE parallel to *dp*. Then take the greateſt and leaſt diſtances of the circle from the pole of projection and ſet from C, to D and F, for the circle DF; and from A, the projecting point, draw AF*f*, and AD*d*, then *df* will be the tranſverſe axis of the ellipſis. But if D fall beyond the line RE, as at G, then draw a line from G backward through A to *d*, and then *df* is the tranſverſe of an hyperbola. But if the point D fall in the line RE as at E, then the line AE no where meets the line of meaſures, and the projection of E is at an infinite diſtance, and then the circle will be projected into a parabola whoſe vertex is *f*. Laſtly, biſect *df* in H the center, and for the ellipſis take half the difference of the lines A*d*, A*f*, and ſet from H to K for the focus. But for the hyperbola take half the ſum of A*d*, A*f*, and ſet from H to the focus *k* of the hyperbola. Then with the tranſverſe *df* and focus K or *k* deſcribe the ellipſis *d*M*f*, or the hyperbola *fm*. For the projection of the circle given. 45.

But for the parabola make EQ = F*f*, and draw *fn* ⊥ AQ, and ſet $\frac{1}{2}$*n*Q from *f* to K the focus. Then with

Fig. with the vertex *f* and focus *k* deſcribe the parabola *fm*, for the projection of the given circle FE.

Otherwiſe by Points.

49. Through the center of projection C, draw the line of meaſures CF, paſſing through the pole P (if P is given; but if not, find it, by ſetting off CP = the diſtance of that pole, from the center of projection, by Prop. XI.) then ſet off PD, PF, equal to the given diſtance from its pole, by Prop. XI. Through P draw a ſufficient number of right lines, Lλ, Mμ, N*n*, O*o*, R*r*, S*s*, &c. which will all repreſent great circles. Find the dividing centers of each of theſe lines; and by Prop. XI. ſet off upon each of them from P, the given diſtance of the circle from its pole, as PL, Pλ, PM, Pμ, &c. and through all the points L, M, D, O, R, &c. draw a curve line, for the circle required.

Note, the dividing center of all the lines, PM, PL, PN, &c. is always diſtant from P the length of PW. Therefore all the dividing centers are in the circumference of a circle, whoſe radius is PW, and center P.; ſee below, Cor. 2.

Or thus.

49. Draw the line of meaſures PCG, and by Prop. XI. make CG = the diſtance of the parallel great circle from the pole of projection, and draw AGK perpendicular to it, which will repreſent a great circle whoſe pole is P. Draw any number of right lines through P to AK, as AP, BP, HP, &c. and by Prop. XI. ſet off from AK the parts AL, BM, HO, &c. each equal to the circle's diſtance from its parallel great circle. Then all the points L, M, D, O, &c. being joined by a regular curve, will repreſent the parallel circle required.

Or

Or thus. Fig.

Through the center of projection C draw the line of meaſures DCF, and the radius of projection CW perpendicular to it, and AGK ⊥ GC, for a great circle whoſe pole is P. Draw *wp* = WP, and *wa* ⊥ to it, draw any number of right lines, AP, BP, GP, &c. and make *pg*, *pb*, *pa*, &c. = PG, PB, PA, &c. alſo make the ∠*pwl* and *pwx* = the circle's diſtance from its pole P (or *awl* = the diſtance from its parallel great circle); and upon PG, PB, PA, &c. make PD, PM, PL, &c. = *pd*, *pm*, *pl*, &c. reſpectively. 49.

Or make GD, BM, AL, &c. = *gd*, *bm*, *al*, &c. After the ſame manner, find the points O, R, &c. and through all the points R, O, D, M, L, &c. draw a regular curve, making no angles, which will repreſent the parallel required. Likewiſe where any line *ap* cuts *wx*, that diſtance from *p* will give the point λ, or *px* = Pλ; and ſo of any other of the lines *bp*, *gp*, &c.

The reaſon of this proceſs will be plain, if you ſuppoſe the point p, w applied to P, W; *and g, b, a, &c. ſucceſſively to* G, B, A, &c. *for then d, m, l, will fall upon* D, M, L, &c.

By the Scale.

Take the tangents of the circle's neareſt and furtheſt diſtance from the pole of projection, and ſet from C to *f* and *d*, gives the vertices, and biſect *df* in H; then take half the difference, or half the ſum, of the ſecants of the greateſt and leaſt diſtance from the pole of projection, and ſet from H, to K or *k* for the focus of the ellipſis or hyperbola, which may then be deſcribed. 45.

Cor. 1. *If the curve be required to paſs through a given point* S; *meaſure* PS *by Prop.* XI, *and then the curve may be drawn by this Problem.* 49.

Cor.

Fig. 53-3. Cor. 2. *If* P *be the point where the great circles* AP, BP, *&c. meet; then to find the dividing centers of all the circles* AP, BP, *&c.*

Through C the center of projection draw PCG. Through C draw CW perpendicular to PG, and equal to the radius of projection. Draw PW, and WG perpendicular to it, to cut GP in G. At G and the middle of GP, erect the perpendiculars GA and DS. With the radius PW deſcribe the circle EWO. Then to find the dividing center of any circle AP; with the diſtance LP, and one foot in L, cut the circle OWE in R, the dividing center required.

For ſince PWG is a right angle, PG is 90 degrees, and ſo is PA. Therefore the dividing center of AP is in a ſemi-circle deſcribed round AP, and it is alſo in the circle OWR (by Cor. 3. Prop. V.) and therefore at the interſection R. But DS biſects all theſe circles; and therefore the center of that ſemi-circle is in L.

PROP. XVI. *Prob.*

47. *To find the pole of any circle in the projection*, DMF.

Rule.

From the center of projection C, draw the radius of projection CA perpendicular to the line of meaſures DF. And to A the projecting point, draw DA, FA, and biſect the angle DAF by the line AP, then P is the pole. But if the curve be an hyperpola, as *fm*, *fig.* 45, you muſt produce *d*A, and biſect the angle *f*AG. And in a parabola, where the point *d* is at an infinite diſtance, biſect the angle *f*AE.

Or thus; Drawing CA perpendicular to DC, draw DA, and make the angle DAP = the circle's diſtance from its poles, give the pole P.

By the Scale. Fig.

Draw the radius of projection CA ⊥ to the line of meaſures DF. Apply CD, CF to the tangents, and ſet the tangent of half the difference of their degrees from C; to P, if D, F lye on contrary ſides of C; but half the ſum if on the ſame ſide, gives P the pole.

Or thus; By Prop. XI. ſet off from D to P, the circle's diſtance from its pole, gives the pole P.

Cor. *If it be a great circle as* BG; *draw the line* 46.
of meaſures GC, *and* CA ⊥ *to it, and equal to the radius of projection*; *make* GAP *a right angle, and* P *is the pole.*

PROP. XVII. *Prob.*

To meaſure any arch of a leſſer circle; *or to ſet any number of degrees thereon.*

Rule.

Let F*n* be the given circle. From the center of 46.
projection C, draw CA perpendicular to the line of meaſures GH, and equal to the radius of projection. To P the pole of the given circle draw AP, and AO biſecting the angle CAP. And draw AD perpendicular to AO. Deſcribe the circle G*l*H (by Prop. XIII.) as far from the pole of projection C, as the given circle is from its pole P. And through any given point *n* in the circle F*n*, draw D*nl*, gives H*l* the number of degrees = F*n*. Or the degrees being given and ſet from H to *l*, the line D*l* cuts off F*n* equal thereto.

Or thus; AO being drawn as before, erect OS perpendicular to CO; through the given point *n* draw P*n* cutting OS in Q, then through Q draw C*l*, and the angle QCP is = F*n*. Or making QCP = the degrees given, draw PQ*n*, and arch F*n* = theſe degrees.

Or

Fig. 46. *Or thus*; AO, AP being drawn as before, draw AG perpendicular to AP, and GB perpendicular to GC. Through the given point *n* draw PB cutting GB in B, and draw OB, then the ∠ GOB = arch F*n* Or making ∠ GOB = the given degrees; draw PB, and it cuts off F*n* = the degrees given.

By the Scale.

Let C be the center of projection, P the pole of the given circle, Apply CP to the tangents, and ſet the tangent of its half from C to O, and the co-tangent of its half from C to D; with radius CG = tangent of the degrees in FP the given circle's diſtance from its pole, deſcribe the circle GSH. Then D*l* drawn through *n* or *l*, cuts off H*l* = F*n*.

Or thus; O being found as before, erect OS perpendicular to CO; through the given point *n* draw PQ*n*, and ∠ QCH = F*n*.

Or thus; Apply CP to the tangents, and ſet the co-tangent thereof from C to G. Erect GB perpendicular to GC. Through *n* draw P*n*B, and draw BO; then ∠ GOB = F*n*.

48. Cor. *If the leſſer circle be perpendicular to the plane of projection as* VHK. *You have no more to do but to draw the perpendiculars* VC, HG, *to its parallel great circle* CI. *Then* CG (*meaſured by Prop.* XI.) *will be equal to* VH; *or the degrees ſet from* C *to* G, *cuts off* VH *equal thereto.*

SCHOLIUM.

48. This ſort of projection is little uſed, by reaſon ſeveral of the circles of the ſphere fall in ellipſes and hyperbolas, which are very difficult to deſcribe. Notwithſtanding it is very convenient for ſolving ſome Problems of the ſphere, becauſe all the great circles are projected into right lines. And this ſort, or the Gnomonic Projection, is the very foundation of all dialling. For if the ſphere be projected on any

any plane, and upon that ſide of it on which the ſun is to ſhine; and the projected pole be made the center of the dial, and the axis of the globe the Stile or Gnomon, and the radius of projection its height; you will have a dial drawn with all its furniture. Upon this account it deſerves to be more taken notice of, than at preſent it is. I have in the foregoing propoſitions given, I think, all the fundamental principles of this kind of projection, having met with little or nothing done upon this ſubject before. Fig.

GENERAL PROBLEM.

To project the ſphere upon any given plane.

Before you can project the ſphere upon any plane, you muſt have a perfect knowledge of all its circles, and their poſitions in reſpect of one another; the diſtances of the leſſer circles from their poles and from their parallel great circles; the angles made by great circles, or their inclinations, to one another, particularly to the primitive circle, on whoſe plane (or a parallel thereto) you are about to project the ſphere. Then, after the primitive circle is deſcribed, you muſt deſcribe all other circles concerned in the Problem, according to the rules of that ſort of Projection, you are going to uſe; and the interſection of theſe circles will determine the Problem.

And note, that the projection of the concave ſide of the ſphere is more fit for aſtronomical purpoſes; for in looking at the heavens, we view the concavity. But it is better to project the convex hemiſphere in geography, becauſe we ſee the convex ſide only.

The principal Points, Angles and Circles of the Sphere are as follows.

I. *Points.*

1. *Zenith* is the point over our heads, Z. 52.

Fig. 53. 55. 2. *Nadir* is the point under our feet, N.

3. *Poles* of the world are two points, round which the diurnal revolution is performed, P the north pole, *p* the ſouth pole. A line drawn through the poles, is called the *Axis* of the world, as P*p*.

4. *The Center* of the earth or of the heavens, C.

5. *Equinoctial Points*, are the points of interſection of the Equator and Ecliptic, ♈, ♎.

6. *Solſtitial Points*, are the beginning of Cancer and Capricorn, ♋, ♑.

II. *Great Circles.*

1. *Equinoctial*, is a circle 90 degrees diſtant from the poles of the world, as EQ.

2. *Meridians*, or *hour Circles*, are circles paſſing through the poles of the world, as P⊙*p*, P*E*p, &c.

3. *Solſtitial Colure*, is a meridian paſſing through the ſolſtitial points, as P♋*p*.

4. *Equinoctial Colure*, is a meridian paſſing through the equinoctial points, PC*p*.

5. *Ecliptic* is the circle through which the ſun ſeems to move in a year, ♋ ♑; it cuts the equinoctial at an angle of 23° 28′, in paſſing through the equinoctial points. In this are reckoned the 12 Signs, ♈, ♉, ♊, ♋, ♌, ♍, ♎, ♏, ♐, ♑, ♒, ♓.

6. *Horizon*, is a circle dividing the upper from the lower hemiſphere, as HO, being 90° diſtant from the Zenith and Nadir.

7. *Vertical Circles*, are circles paſſing through the Zenith and Nadir, Z⊙N.

8. *Circles of Longitude* in the heavens, paſs through the poles of the ecliptic and cut it at right angles.

9. *Meridian of a Place*, is that Meridian which paſſes through the Zenith, as PZH.

10. *Prime Vertical*, is that which paſſes through the eaſt and weſt points of the horizon.

III. *Leſſer circles.*

1. *Parallels of Latitude* are parallel to the equinoctial on the earth, *parallels of altitude* are parallel

to

to the horizon, *parallels of declination* are parallel to the equinoctial in the heavens. Fig. 52.

2. *Tropics*, are two circles distant 23° 28′ from the equinoctial, the tropic of *Cancer* towards the north, the tropic of *Capricorn* towards the south. 53. 55.

3. *Polar Circles*, are distant 23° 28′ from the poles of the world, the *Arctic* circle towards the north, the *antarctic* towards the south.

IV. *Angles and Arches of Circles.*

1. *Sun's* (or *Star's*) *Altitude*, is an arch of the Azimuth between the sun and horizon, as ⊙ B.

2. *Amplitude* is an arch of the horizon, between sun-rising and the east, or sun-setting and the west.

3. *Azimuth*, is an arch of the horizon between the sun's Azimuth circle, and the north or south, as HB, or OB, or it is the angle at the zenith, HZB or OZB.

4. *Right Ascension* is an arch of the equator between the sun's meridian, and the first point of Aries, as ♈ K.

5. *Ascensional Difference* is an arch of the equinoctial, between the sun's meridian, and that point of the equinoctial that rises with him, or it is the angle at the pole between the sun's and the six o'clock meridian.

6. *Obilque Ascension* or *Descension*, is the sum or difference of the right ascension and the ascensional difference.

7. *Sun's Longitude*, is an arch of the ecliptic, between the sun and first part of Aries, as ♈ ⊙.

8. *Declination* is an arch of the meridian, between the equinoctial and the sun, as ⊙K.

9. *Latitude of a Star*, is an arch of a circle of longitude between the star and ecliptic.

10. *Latitude of a place*, is an arch of the meridian between the equinoctial and the place.

11. *Longitude* of a place on the earth is an arch of the equinoctial, between the first meridian (Isle of *Ferro*), and the meridian of the place. And

Fig. 52. 53. 55. *diff. longitude*, is an arch of the equator, between the meridians of the two places, or the angle at the pole.

12. *Hour of the Day*, is an arch of the equinoctial, between the meridian of the place and the ſun's meridian, as EK; or it is the angle they make at the pole, as EPO.

Example I.

To project the ſphere upon the plane of the meridian, for May 12, 1767. *Latitude* 54° ½ *north, at a quarter paſt* 9 *o'clock before noon.*

I. *By the Orthographic Projection.*

52. Here we will project the convex ſide of the eaſtern hemiſphere. With the chord of 60 degrees deſcribe the primitive circle or meridian of the place HZON. Through the center C draw the horizon HO; ſet the latitude 54½ from O to P and from H to *p*, and draw P*p* the 6 o'clock meridian. Through C draw EQ perpendicular to P*p* for the equinoctial. Make ED, Q*d* 18° 5′ the declination May 12, and draw D*d* the ſun's parallel for that day. By Prop. XI. make ⊙ G (3¼ hours or) 48° 45′ the ſun's diſtance from the hour of 6, then ⊙ is the ſun's place. Through ⊙ by Prop. V. draw AL parallel to H ⊙ for the ſun's parallel of altitude. By Prop. VII. draw the meridian P ⊙ *p* and the azimuth Z⊙N. Alſo the ecliptic will be an ellipſis paſſing through ⊙, which cannot conveniently be drawn in this projection. Alſo draw the parallel S*s* 18° below the horizon, and where it interſects D*d* is the point of day break, if there is any. Now the ſun is at *d* at 12 o'clock at night, and riſes at R, at 6 o'clock is at G, due eaſt at F, at ⊙ a quarter paſt 9, and is at D in the meridian at 12 o'clock.

Draw GI parallel to HO. Then GR meaſured by Prop. X. is 27° 14′, and turned into time (allowing 15 degrees for an hour) ſhows how long the ſun riſes before 6, to be 1ʰ 49ᵐ; GI meaſured by Prop. X. gives the azimuth at 6, 79° 16′. CR by Cor. Prop.

Prop. X. gives the amplitude 32° 19′, and CF gives his altitude when east 22° 25′. FG 13° 28′ (turned into time) is 54^{m}, and shews how long after 6 he is due east. IO is his altitude at 6, 14° 38′. AH 41° 53′ is his altitude at ⊙, or a quarter past 9; and ⊙L measured by Prop. X. is his azimuth from the north at the same time, 122° 40′. And thus the place of the moon or star being given, it may be put into the projection, as at ☽. And its altitude, azimuth, amplitude, time of rising, &c. may all be found, as before for the sun. Fig. 52.

II. *Stereographically.*

To project the sphere on the plane of the meridian, the projecting point in the western point of the horizon; with chord of 60, draw the primitive circle HZON, and through C draw HO for the horizon, and ZN perpendicular thereto for the prime vertical. Set the latitude from O to P, and from H to *p*, and draw P*p* the 6 o'clock meridian, and EQ perpendicular thereto for the equinoctial. Make ED, Q*d* the declination, and by Prop. XII. draw DG*d*, the sun's parallel for the day. Draw the meridian P⊙*p* by Prop. XVII making an angle of 41° 15′ with the primitive, to intersect the sun's parallel in ⊙, the sun's place at 9$^{h}\frac{1}{4}$. Through ⊙, by Prop. XII. draw the parallel of altitude A⊙L; through ⊙ draw, by Prop. XVII. the azimuth Z⊙N. And by Prop XII. draw the parallel S*sd* 18° below the horizon, if it cut R*d*, gives the point of day break. And through G draw the parallel of altitude GI. Lastly, by Prop. XX. through ⊙ draw the great circle ♈⊙♎ cutting the equinoctial EQ at an angle of 23° 28′, and this is the ecliptic, ♈ the first point of Aries, and ♎ that of Libra. 53.

This done, *d*R measured by Prop. XXIII. is 62° 46′, shows the time of sun rising; CR by Prop. XXII. is the amplitude 32° 19′. GI 79° 16′ by Prop. XXIII. the sun's azimuth at 6. IO 14° 38′ his

Fig. 53. his altitude at 6. CF 22° 25′ by Prop. XXII. his altitude when east. GF 13° 28′ the time when he is due east. ⊙B 41° 53′ by Prop. XXII. his altitude at a quarter past 9; the ∠⊙ZP 122° 40′ by Prop. XXIV. his azimuth at that time. Also ♈⊙, by Prop. XXII. is his longitude 51° 7′. ♈K his right ascension, 48° 40′.

And the place of the moon or a star being given, it may be put into the scheme as at ✳; and its time of rising, amplitude, azimuth, &c. found as before.

III. *Gnomonically.*

54. To project the eastern hemisphere upon a plane parallel to the meridian. About the center of projection C describe the circle HON with the tangent of 45 the radius of projection, for the primitive. Through C draw the horizon HO, and the prime vertical ZN perpendicular thereto. Set the latitude 54½ from H to *a*, and draw the 6 o'clock meridian P*p*, and the equinoctial EQ perpendicular to it. Set the tangent of 48° 45′ (equal to 3¼ hours) from C to E, and by Prop. X. draw the meridian EL parallel to P*p*. Make E*e* = E*a*, and ∠E*e*⊙ = 18° 5′ the sun's declination, then by Prop. XI. ⊙ is the sun's place. Through ⊙ draw the hyperbola D⊙*d* (by Prop. XIV.) for the sun's parallel of declination; and draw ⊙B perpendicular to HO, for his azimuth circle. And draw GI perpendicular to HO, and RM, FT, ‖ P*p*. Also the ecliptic is a right line passing through ⊙, and cutting EQ at an angle of 23° 28′, which is difficult to draw in this projection.

Also by Prop. XIV. Draw the parallel S*s* 18° below the horizon, and if it intersects D*d*, it gives the point of sun rise.

Then if by Prop. XVII. or XI. you measure GR or rather CM, 27° 14′, you have the time of sun rising; GF or CT 13° 28′, the time when he is due east. Also by Prop. XI. if you measure CR, you have

have the amplitude 32° 19′. CI the comp. of his azimuth at ſix, 10° 44′. IG by Prop. XII. his altitude at ſix, 14° 38′. CF his altitude when eaſt, 22° 25′. And by Prop. XI. ⊙B = 41° 53′, his altitude a quarter paſt nine. CB the complement of his azimuth at that time 32° 40′.. Fig. 54.

And the place of the moon or a ſtar being given, its place in the projection may be determined as before, and all the requiſites found.

Ex. 2.

To project the ſphere upon the plane of the ſolſtitial colure for latitude 54½ *N. May* 23, 1767, *at* 10 *o'clock in the morning.*

Stereographically.

The projection of the weſtern hemiſphere, the firſt point of Libra, the projecting point. Deſcribe the ſolſtitial colure PE*p*Q, and the equinoctial colure P*p* perpendicular to it; and through C draw the equinoctial EQ perpendicular to P*p*. Set 23° 28′ from E to ♋, and from Q to ♑, and draw the ecliptic ♋ ♑. Set the ſun's longitude 61° 42′ from C to ⊙, and through ⊙ draw P⊙K*p* for the 10 o'clock meridian. Make KA (two hours or) 30°, and draw PA*p* for the meridian of the place. Set the latitude of the place 54½ from A to Z, and Z is the zenith. About the pole Z deſcribe the great circle BHS for the horizon of the place. Through Z and ⊙ draw an azimuth circle Z⊙B. 55.

Then you have ⊙K the ſun's declination 20° 33′. CK his right aſcenſion 59° 35′. ⊙B his altitude at 10 o'clok 49° 10′; the ∠AZ⊙ or PZ⊙ his azimuth at 10 = HB, 45° 44′. H the ſouth point of the horizon. I the point of the ecliptic that is in the meridian. T the point of the ecliptic that is ſetting in the horizon.

Example 3.

To project the ſphere on the plane of the horizon, Lat. 35½, *N. July* 31, 1767, *at* 10 *o'clock.*

Fig. *Gnomonically.*

56. To project the upper hemiſphere on a plane parallel to the horizon. With the radius of projection and center C, deſcribe the primitive circle ADB. Through C draw the meridian PE, and AS perpendicular to it for the prime vertical. Set off CP $35\frac{1}{2}$ the latitude and P is the N. Pole, and perpendicular to CP draw Pp the 6 o'clock meridian. Set the complement of the latitude from C to E; and draw EQ perpendicular to CE for the equinoctial. Make EB 30° (or 2 hours) and draw the 10 o'clock meridian PB. Set the ſun's declination 18° 27′ from B to ⊙. And ⊙ is the place of the ſun at 10 o'clock. Through ⊙ draw the azimuth circle CQ; likewiſe through ⊙, a parallel to the equinoctial EQ may eaſily be deſcribed by Prop. XV. for the ſun's parallel that day.

Then C⊙ meaſured by Prop. XI. is 31° 30′ the complement of the altitude. And the angle EC⊙ meaſured by Cor. Prop. XII. is his azimuth, 65° 10′.

Scholium.

After this manner may any Problems of the Sphere be ſolved by any of theſe projections, or upon any planes, but upon ſome more commodiouſly than upon others. And if in a ſpherical triangle any ſides or angles be required, they may be projected from what is given therein, according to any of theſe kinds of projection before delivered; and it will be moſt eaſily done, when you chuſe ſuch a plane to project on, that ſome given ſide may be in the primitive, or a given angle at the center; and then you need draw no more lines or circles than what are immediately concerned in that Problem. But always chuſe ſuch a plane to project on, where the lines and circles are moſt eaſily drawn, and ſo that none of them run out of the ſcheme.

THE

LAWS

OF

Centripetal and Centrifugal Force.

SHEWING

The Motion of Bodies in Circular Orbits, and in the Conic Sections, and other Curves.

And explaining the perturbating Force of a third Body. With many other things of the like Nature.

Being a Work preparatory to Astronomy, and the very basis thereof. And absolutely necessary to be known by all such as desire to be Proficients in that Science.

Solis uti varios cursus, lunæque meatus
Noscere possemus, quæ vis, & causa cieret.

Lucret. Lib. V.

THE

PREFACE.

IN the following Treatiſe, I have explained and demonſtrated the Laws of Centripetal Forces; a doctrine upon which all Aſtronomy is grounded; and without the knowledge of which, no rational account can be given of the motions of any of the celeſtial bodies, as the Comets, the Planets, and their Satellites. From theſe laws are derived the cauſes of the various ſeeming irregularities obſerved in their motions; ſuch as their accelerations and retardations, their approaching to, and receding from the center of force; irregularities only in appearance; but in reality, theſe motions are truly regular and conformable to the eſtabliſhed laws of Nature. From this foundation we trace the way or path of all the planets, and diſcover the origin and ſpring of all the celeſtial motions, and clearly underſtand and account for all the phænomena thence ariſing.

In the firſt ſection, you have the Centripetal Forces of bodies revolving in circles; their velocities, periodic times, and diſtances compared together; their relations and proportions to each other; and that when they either revolve about the ſame center, or about different ones. The different motions cauſed by different forces, or by different central attracting bodies, are here ſhewn. We have given likewiſe the periodic time of a ſimple pendulum revolving with a conical motion; and alſo the center of Turbination, and the periodic time of a compound pendulum, or a ſyſtem of bodies, revolving with a conical motion; as properly belonging to the doctrine of Centripetal Forces.

In the ſecond ſection we have ſhewn the motion of bodies in the Ellipſis, Hyperbola, and Parabola; and in other Curves. The proportion of the Centripetal Forces, and velocities in different parts of the ſame Curve. The law of Centripetal Force to deſcribe a given Curve, and the velocity

in

in any point of it; and more particularly with reſpect to that law of Centripetal Force that is reciprocally as the ſquare of the diſtance; which is the grand law of Nature in regard to the action of bodies upon one another at a diſtance; and according to this law, is ſhewn the motion of bodies round one another, and round the common center of gravity, and the orbits they will deſcribe.

In the third ſection we have given the diſturbing or perturbating force of a third body, acting upon two others that revolve round one another. From theſe principles are deduced the errors cauſed in the motion of a Satellite, moving round its primary planet. Towards the end, are ſeveral propoſitions, by means whereof, the motion of the Nodes, and variation of inclination of a Satellite's orbit, and ſuch like things may be computed. As theſe things are all laid down for the ſake of underſtanding our own Syſtem, I have inſerted ſome few things, by way of illuſtration of the rules, in regard to the Moon and Jupiter. But as to the Moon, there are ſome things ſo very intricate, and require ſuch long and tedious calculations, as would require a volume of themſelves; ſo that the ſmall room I am confined to cannot admit of them; and few would trouble themſelves to read them, if they were there. This laſt ſection concludes with a few things of another kind, but depending on the principles of Centripetal Forces.

Several of thoſe things about Centripetal Forces are calculated by the method of Fluxions; and cannot eaſily be done any other way; and moſt of them taken from my book of Fluxions. And ſeveral other things relating to Centripetal Forces, you will alſo find in that book; being ſorry to trouble the reader too much with repeating what I have written and publiſhed elſewhere.

W. Emerſon.

THE LAWS OF Centripetal and Centrifugal Force.

DEFINITIONS.

DEF. I.

THE center of attraction, is the point towards which any body is attracted or impelled.

DEF. II.

Centripetal force, is that force by which a body is drawn or impelled to a certain point, as a center. Here all the particles of the body are equally acted on by the force.

DEF. III.

Centrifugal force, is the reſiſtance a moving body makes to prevent its being turned out of its direct courſe. This is oppoſite and equal to the centripetal force; for action and re-action are equal and contrary.

DEF. IV.

Angular velocity, is the quantity of the angle a body deſcribes in a given time, about a certain point, as a center. *Apparent velocity* is the ſame thing.

DEF. V.

Periodical time, is the time of revolution of a body round a center.

SECT.

SECT. I.

The Motion of Bodies in Circular ORBITS.

PROP. I.

Fig. *THE centripetal forces, whereby equal bodies at equal diſtances from the centers of force, are drawn towards theſe centers; are as the quantities of matter in the central bodies.*

For ſince all attraction is made towards bodies, every part of the attracting body muſt contribute its ſhare in that effect. Therefore a body twice as great will attract the ſame body twice as much; and one thrice as great, thrice as much, and ſo on. Therefore the attraction of the central body; that is, the centripetal force, is as the quantity of matter in the attracting or central body.

Cor. 1. *Any body whether great or little, placed at the ſame diſtance, is attracted through equal ſpaces in the ſame time, by the central body.*

For though a body twice or thrice as great as another, is drawn with twice or thrice the force; yet it will acquire no greater velocity, nor paſs through a greater ſpace. For (Prop. V. Cor. 2. Mechan.) the velocity generated in a given time, is as the force directly, and quantity of matter reciprocally; and the force, which is the weight of the body, being as the quantity of matter; therefore the velocity generated is as the quantity of matter directly, and quantity of matter reciprocally, and therefore is a given quantity.

Cor.

Cor 2. *Therefore the centripetal force, or force towards the center, is not to be meaſured by the quantity of the falling body; but by the ſpace it falls through in a given time. And therefore it is ſometimes called an accelerative force.* Fig.

PROP. II.

If a body revolves in a circle, and is retained in it, by a centripetal force, tending to the center of it; put 1.

R = *radius of the circle or orbit deſcribed,* AC.

F = *abſolute force, at the diſtance* R.

s = *the ſpace a falling body could deſcend through, by the force at* A, *and*

t = *time of the deſcent.*

π = 3.1416.

Then its periodic time, or the time of one revolution, will be $\pi t\sqrt{\frac{2R}{s}}$.

And the velocity, or ſpace it deſcribes in the time t, will be $\sqrt{2Rs}$.

For let AB be a tangent to the circle at A; take AF an infinitely ſmall arch, and draw FB perp. to AB, and FD perp. to the radius AC. Let the body deſcend through the infinitely ſmall height AD or BF, by the centripetal force in the time 1. Now that the body may be kept in the circular orbit AFE, it ought to deſcribe the arch AF in the ſame time 1. The circumference of the circle AE is 2πR, and the arch AF = $\sqrt{2R \times AD}$.

By the laws of falling bodies $\sqrt{s} : t :: \sqrt{AD} : t\sqrt{\frac{AD}{s}}$ = time of moving through AD or AF. And by uniform motion, as AF, to the time of its deſcription : : circumference AFEA, to the time of one

Fig. 1. one revolution; that is, $\sqrt{2R \times AD} : t\sqrt{\frac{AD}{s}} ::$

$2\pi R$: periodic time $= \frac{2t\pi R}{\sqrt{2Rs}} = \pi t\sqrt{\frac{2R}{s}}$.

Alſo by the laws of uniform motion, $t\sqrt{\frac{AD}{s}}$ or time of deſcribing AF : AF or $\sqrt{2R \times AD} ::$ $t : \sqrt{2Rs}$ = the velocity of the body, or ſpace deſcribed in time t.

Cor. 1. *The velocity of the revolving body, is equal to that which a falling body acquires in deſcending through half the radius* AC, *by the force at* A *uniformly continued.*

For $\sqrt{s}$ (height) : $2s$ (the velocity) : : $\sqrt{\frac{1}{2}}$R (the height) : $\sqrt{2Rs}$, the velocity acquired by falling through $\frac{1}{2}$R.

Cor. 2. *Hence, if a body revolves uniformly in a circle, by means of a given centripetal force; the arch which it deſcribes in any time, is a mean proportional between the diameter of the circle, and the ſpace which the body would deſcend through in the ſame time, and with the ſame given force.*

For 2R (diameter) : $\sqrt{2Rs} :: \sqrt{2Rs} : s$; where $\sqrt{2Rs}$ is the arch deſcribed, and s the ſpace deſcended through, in the time t.

2. Cor. 3. *If a body revolves in any curve* AFQ, *about the center of force* S; *and if* AC *or* R *be the radius of curvature in any point* A; s = *ſpace deſcended by the force directed to* C. *Then the velocity in* A *will be* $\sqrt{2Rs}$.

For this is the velocity in the circle; and therefore in the curve, which coincides with it.

PROP.

Fig.

PROP. III.

If several bodies revolve in circles round the same or different centers; the periodic times will be as the square roots of the radii directly, and the square roots of the centripetal forces reciprocally. 1.

Let F = centripetal force at A tending to the center C of the circle.
V = velocity of the body.
R = radius AC of the circle.
P = periodic time.

Then (Prop. II.) $P = \pi t \sqrt{\frac{2R}{s}}$. But s is as the force F that generates it; whence $P = \pi t \sqrt{\frac{2R}{F}}$, and since 2, π and t are given quantities, therefore $P \propto \sqrt{\frac{R}{F}}$.

Cor. 1. *The periodic times are as the radii directly, and the velocities reciprocally.*

For (Prop. II.) $V = \sqrt{2Rs} = \sqrt{2RF}$, and $V^2 = 2RF$, and $P = \pi t \sqrt{\frac{2R}{F}}$, and $PP = \pi^2 t^2 \times \frac{2R}{F}$, therefore $P^2 V^2 = \pi^2 t^2 \times 4R^2$, and $P^2 = \frac{\pi^2 t^2 \times 4R^2}{V^2}$, and $P = \frac{\pi t \times 2R}{V} \propto \frac{R}{V}$.

Cor. 2. *The periodic times are as the velocities directly, and the centripetal forces reciprocally.*

For $V^2 = 2Rs = 2RF$; and $R = \frac{VV}{2F}$, and $\frac{R}{V} = \frac{V}{2F} \propto \frac{V}{F}$. But (Cor. 1.) $P \propto \frac{R}{V} \propto \frac{V}{F}$.

Fig. 1. Cor. 3. *If the periodic times are equal; the velocities, and alſo the centripetal forces, will be as the radii.*

For if P be given; then $\frac{R}{F}$, and $\frac{R}{V}$, and $\frac{V}{F}$ are all given ratios.

Cor. 4. *If the periodic times are as the ſquare roots of the radii; the velocities will be as the ſquare roots of the radii, and the centripetal forces equal.*

For (Prop. III. and Cor. 1.) putting $\sqrt{R}$ for P, we have $\sqrt{R} \propto \sqrt{\frac{R}{F}} \propto \frac{R}{V}$. Therefore $1 \propto \frac{1}{\sqrt{F}} \propto \frac{\sqrt{R}}{V}$, and $\sqrt{R} \propto V$, and $\sqrt{F}$ is a given quantity.

Cor. 5. *If the periodic times are as the radii; the velocities will be equal, and the centripetal forces reciprocally as the radii.*

For putting R for P, we have $R \propto \sqrt{\frac{R}{F}} \propto \frac{R}{V}$; whence $\sqrt{R} \propto \frac{1}{\sqrt{F}}$, and $1 \propto \frac{1}{V}$; that is, $R \propto \frac{1}{F}$, or the centripetal force is reciprocally as the radius; and V is a given quantity.

Cor. 6. *If the periodic times are in the ſeſquiplicate ratio of the radii; the velocities will be reciprocally as the ſquare roots of the radii, and the centripetal forces reciprocally as the ſquares of the radii.*

Put R^3 for P, then $R^{\frac{3}{2}} \propto \sqrt{\frac{R}{F}} \propto \frac{R}{V}$; and $R \propto \frac{1}{\sqrt{F}}$ or $RR \propto \frac{1}{F}$, and $\sqrt{R} \propto \frac{1}{V}$.

Cor. 7. *If the periodic times be as the nth power of the radius; then the velocities will be reciprocally as the $n-1^{th}$ power of the radii, and the centripetal forces*

forces reciprocally as the $2n-1^{th}$ *power of the radii.* Fig. 1.

Put R^n for P, then $R^n \propto \sqrt{\frac{R}{F}} \propto \frac{R}{V}$. Whence $R^{2n} \propto \frac{R}{F}$, and $R^{2n-1} \propto \frac{1}{F}$. Alſo $R^{n-1} \propto \frac{1}{V}$.

PROP. IV.

If ſeveral bodies revolve in circles round the ſame or different centers; the velocities are as the radii directly, and periodic times reciprocally. 1.

For putting the ſame letters as in Prop. III. we have (by Prop. II.) $V = \sqrt{2Rs} = \sqrt{2RF}$; and $P \propto \frac{V}{F}$ (by Cor. 2. Pr. III.) and $PF \propto V$, and $F \propto \frac{V}{P}$. Whence $V = \sqrt{2RF} = \sqrt{2R \times \frac{V}{P}}$, and $V^2 = \frac{2RV}{P}$, and $V = \frac{2R}{P} \propto \frac{R}{P}$.

Cor. 1. *The velocities are as the periodical times, and the centripetal forces.*

For we had $PF \propto V$.

Cor. 2. *The ſquares of the velocities are as the radii and the centripetal forces.*

For $V = \sqrt{2RF}$.

Cor. 3. *If the velocities are equal; the periodic times are as the radii, and the radii reciprocally as the centripetal forces.*

For if V be given, its equal $\frac{R}{P}$ is a given ratio; and $\sqrt{RF}$ is given, whence $R \propto \frac{1}{F}$.

Fig. 1. Cor. 4. *If the velocities be as the radii, the periodic times will be the ſame; and the centripetal forces as the radii.*

For then V or $R \propto \frac{R}{P}$, and $1 \propto \frac{1}{P}$. Alſo $R = \sqrt{2RF}$, whence $R \propto F$.

Cor. 5. *If the velocities be reciprocally as the radii, the centripetal forces are reciprocrlly as the cubes of the radii; and the periodic times as the ſquares of the radii.*

For put $\frac{1}{R}$ for V, then (Cor. 2.) $\frac{1}{R} = \sqrt{2RF}$, $\frac{1}{RR} = 2RF$, whence $F \propto \frac{1}{R^3}$. Alſo $\frac{1}{R} \propto \frac{R}{P}$, and $P \propto RR$.

PROP. V.

1. *If ſeveral bodies revolve in circles about the ſame or different centers; the centripetal forces are as the radii directly, and the ſquares of the periodic times reciprocally.*

Put the ſame letters as in Prop. III. Then (Prop. II.) $P = \pi t \sqrt{\frac{2R}{s}} = \pi t \sqrt{\frac{2R}{F}}$, and $PP = \pi\pi tt \times \frac{2R}{F}$, and $PPF = 2\pi\pi tt R$; whence $F = \frac{2\pi\pi tt R}{PP} \propto \frac{R}{PP}$.

Cor. 1. *The centripetal forces, are as the velocities directly, and the periodic times reciprocally.*

For (Prop. IV.) $V \propto \frac{R}{P}$, and $F \propto \frac{R}{PP} \propto \frac{V}{P}$.

Cor. 2. *The centripetal forces, are as the ſquares of the velocities directly, and the radii reciprocally.*

For

For (Cor. 1.) $F \propto \frac{V}{P}$, and $FP \propto V$. But (Prop. III. Cor. 1.) $P \propto \frac{R}{V}$, therefore $FP \propto \frac{FR}{V}$, therefore $\frac{FR}{V} \propto V$, and $F \propto \frac{VV}{R}$. Fig. 1.

Cor. 3. *If the centripetal forces are equal; the velocities are as the periodic times; and the radii as the ſquares of the periodic times, or as the ſquares of the velocities.*

Cor. 4. *If the centripetal forces be as the radii, the periodic times will be equal.*

For $F \propto \frac{R}{PP}$, and $\frac{F}{R} \propto \frac{1}{PP}$, and if $\frac{F}{R}$ be a given ratio, $\frac{1}{PP}$ will be given, as alſo P.

Cor 5. *If the centripetal forces be reciprocally as the ſquares of the diſtances; the ſquares of the periodical times will be as the cubes of the diſtances; and the velocities reciprocally as the ſquare roots of the diſtances.*

For writing $\frac{1}{RR}$ for F, then $\frac{1}{RR} \propto \frac{R}{PP}$, and $\frac{R^3}{P^2}$ a given quantity. And $\frac{1}{RR} \propto \frac{VV}{R}$, and $\frac{1}{R} \propto VV$, or $\sqrt{\frac{1}{R}} \propto V$.

PROP. VI.

If ſeveral bodies revolve in circles, about the ſame or different centers; the radii are directly as the centripetal forces, and the ſquares of the periodic times. 1.

For (Prop. II.) putting the ſame letters as before, $P = \pi t \sqrt{\frac{2R}{s}} = \pi t \sqrt{\frac{2R}{F}}$, and $PP = \pi\pi tt \times \frac{2R}{F}$, and $PPF = 2\pi\pi ttR \propto R$.

Fig. 1. Cor. 1. *The radii are directly as the velocities and periodic times.*

For (Prop. IV. Cor. 1.) $PF \propto V$, but $PPF \propto R$; therefore $PV \propto R$.

Cor. 2. *The radii are as the ſquares of the velocities directly, and the centripetal forces reciprocally.*

For (Prop. III. Cor. 2.) $P \propto \frac{V}{F}$, but (Cor. 1.) $R \propto PV$; therefore $R \propto \frac{VV}{F}$.

Cor. 3. *If the radii are equal; the centripetal forces are as the ſquares of the velocities, and reciprocally as the ſquares of the periodic times. And the velocities reciprocally as the periodic times.*

For if R be given, $\frac{VV}{F}$, and PPF, and PV, are given quantities, and $F \propto VV$, or $F \propto \frac{1}{PP}$, and $V \propto \frac{1}{P}$

Scholium.

The converſe of all theſe propoſitions and corollaries are equally true. And what is demonſtrated of centripetal forces, is equally true of centrifugal forces, they being equal and contrary.

PROP. VII.

1. *The quantities of matter in all attracting bodies, having others revolving about them in circles; are as the cubes of the diſtances directly, and the ſquares of the periodical times reciprocally.*

Let M be the quantity of matter in any central attracting body. Then ſince it appears, from all aſtronomical obſervations, that the ſquares of the periodical times are as the cubes of the diſtances, of the planets, and ſatellites from their reſpective centers

centers. Therefore (Cor. 6. Prop. III.) the centripetal forces will be reciprocally as the ſquares of the diſtances; that is, $F \propto \frac{1}{RR}$. And (Prop. I.) the attractive force at a given diſtance, is as the body M, therefore the abſolute force of the body M is as $\frac{M}{RR}$. And (Prop. V.) ſince $F \propto \frac{R}{PP}$, put $\frac{M}{RR}$ inſtead of F, and we have $\frac{M}{RR} \propto \frac{R}{PP}$, and $M \propto \frac{R^3}{P^2}$. Fig. 1.

Cor. 1. *Hence inſtead of* F *in any of the foregoing propoſitions and their corollaries, one may ſubſtitute* $\frac{M}{RR}$, *which is the force that the attracting body in* C, *exerts at* A.

Cor. 2. *The attractive force of any body, is as the quantity of matter directly, and the ſquare of the diſtance reciprocally.*

PROP. VIII.

If the centripetal force be as the diſtance from the center C. *A body let fall from any point* A, *will fall to the center in the ſame time, that a body revolving in the circular orbit* ALEA, *at the diſtance* CA, *would deſcribe the quadrant* AGL. 3.

The truth of this is very readily ſhewn by fluxions; thus, put $AC = r$, $AH = x$, t = time of deſcribing AH, v = the velocity at H. F = force at H, which is as CH or $r - x$. Then (Mechan. Cor. 2. Prop. V.) the velocity generated is as the force and time; that is, $\dot{v} \propto F\dot{t}$. Alſo (Mechan. Prop. III. Cor. 1.) the time is as the ſpace divided by the veloci-

Fig. 3. ty; that is, $\dot{t} \propto \frac{\dot{x}}{v}$; therefore $\dot{v} \propto \frac{F\dot{x}}{v} \propto \frac{\overline{r-x} \times \dot{x}}{v}$, and $v\dot{v} \propto \overline{r-x} \times \dot{x}$, and the fluent is $\frac{vv}{2} \propto rx - \frac{xx}{2}$, or $vv \propto 2rx - xx$, and $v \propto \sqrt{2rx - xx}$ or HG; that is, the velocity at H is as the ordinate HG of the circle.

Now it is evident, that in the time the revolving body deſcribes the infinitely ſmall arch AF, the falling body will deſcend through the verſed ſine AD, and would deſcribe twice AD in the ſame time, with the velocity in D. Therefore we ſhall have,

velocity at F : velocity at D : : AF or FD : 2AD,
and velocity at D : velocity at H : ; AF or FD : GH,

therefore,

velocity at F or G : velocity at H : : $AF^2 : 2AD \times GH :: \frac{AF^2}{2AD} : GH$: : CA or CG : GH. But drawing an ordinate infinitely near GH; by the nature of the circle, it will be, as GC : GH : : ſo the increment of the curve AG : to the increment of the axis AH. And Therefore, vel. at G : vel. at H : : as the increment of AG : to the increment of AH. Therefore ſince the velocities are as the ſpaces deſcribed, the times of deſcription will be equal; and the ſeveral parts of the arch AGL are deſcribed in the ſame times as the correſpondent parts of the radius AHC. And by compoſition, the arch AG and abſciſſa AH, as alſo the quadrant AL and radius AC, are deſcribed in equal times.

Cor. 1. *The velocity of the deſcending body at any place* H, *is as the ſine* GH.

Cor. 2. *And the time of deſcending through any verſed ſine* AH, *is as the correſpondent arch* AG.

Cor.

Cor. 3. *All the times of falling from any altitudes whatever, to the center* C, *will be equal.* Fig. 3.

For theſe times are $\frac{1}{4}$ the periodic times; and (Prop. V. Cor. 4.) theſe periodic times are all equal.

Cor. 4. *In the time of one revolution, the falling body will have moved from* C *through* E, *and back again through* C *to* A, *meeting the revolving body again at* A.

Cor. 5. *The velocity of the falling body at the center* C, *is equal to the velocity of the revolving body.*

For the velocities are as the lines GH and GC; and theſe are equal, when G comes to L.

PROP. IX.

If a pendulum AB *be ſuſpended at* A, *and be made to revolve by a conical motion, and deſcribe the circle* BEDH *parallel to the horizon.* 4.

Put $\pi = 3.1416$; $p = 16\frac{1}{2}$ *feet, the ſpace deſcended by gravity in the time* t.

Then the periodical time of B *will be* $\pi t \sqrt{\frac{2AC}{p}}$.

For (Mechan. Prop. VIII.) if the axis AC repreſents the weight of the body, AB will be the force ſtretching the ſtring, and BC the force tending to the center C. Alſo (Mechan. Prop. VI.) if the time is given, the ſpace deſcribed will be as the force; whence $AC : BC :: p : \frac{BC}{AC}p$ = the ſpace deſcended towards C, by the force BC, in the time t. This is the ſpace s in Prop. II. Therefore inſtead of s put its value in the periodical time, and (by Prop. II.) we ſhall have the periodical time of the pendulum $= \pi t \sqrt{\frac{2R}{s}} = \pi t \sqrt{2BC \times \frac{AC}{BC \times p}}$

$= \pi t \sqrt{\frac{2AC}{p}}$.

Cor.

Fig. 4. Cor. 1. *In all pendulums, the periodic times are as the ſquare roots of the heights of the cones,* AC.

For π, t, and p are given quantities.

Cor. 2. *If the heights of the cones be the ſame, the periodic times will be the ſame, whatever be the radius of the baſe* BC.

Cor. 3. *The ſemiperiodic time of revolution, is equal to the time of oſcillation of a peudulum, whoſe length is* AC, *the height of the cone.*

For by the laws of falling bodies, $t\sqrt{\frac{AC}{2p}}$ = time of falling through $\frac{1}{2}$AC; and therefore (Mechan. Prop. XXIV.) $1 : \pi :: t\sqrt{\frac{AC}{2p}} : \pi t\sqrt{\frac{AC}{2p}}$ $= \frac{1}{2}\pi t\sqrt{\frac{2AC}{p}}$, the time of vibration, which is half the periodical time.

Cor. 4. *The ſpace deſcended by a falling body in the time of one revolution, will be* $\pi\pi \times 2$AC.

For tt (time) : p (height) : : $\pi\pi tt \times \frac{2AC}{p}$ (per. time) : $\pi\pi \times 2$AC = height deſcended in that time.

Cor. 5. *The periodic time, or time of one revolution, is equal to* $\pi\sqrt{2} \times$ *time of falling through* AC.

For the time of falling through AC is $t\sqrt{\frac{AC}{p}}$.

Cor. 6. *The weight of the pendulum is to the centrifugal force; as the height of the cone* AC, *to the radius of the baſe* CB. *And therefore when the height* CA *is equal to the radius* CB; *the centripetal or centrifugal force is equal to the gravity.*

PROP.

PROP. X.

Fig. 5.

Suppoſe a ſyſtem of bodies A, B, C, *to revolve with a conical motion about the axis* TR *perp. to the horizon, ſo as to keep the ſame ſide always towards the axis of revolution, and the ſame poſition among themſelves.*

To find the periodical time of the whole ſyſtem.

1. Let A, B, C, be all ſituated in one plane paſſing through TR. From A, B, C let fall the perpendiculars A*a*, B*b*, C*c*, upon the axis TR. And let A, B, C repreſent the quantities of matter in the bodies A, B, C. Alſo put $h = 16\frac{1}{12}$ feet the height a body falls in the time t by gravity; $\pi = 3.1416$; P = the periodic time of the ſyſtem.

By the reſolution of forces, T*a* (gravity) : A*a* (force in direction A*a*) : : $h : \frac{Aa}{Ta}h$ = ſpace deſcended by A towards a in the time 1, which is as the velocity generated by the force A*a*. Therefore $\frac{Aa}{Ta}h$A = motion generated in A in direction A*a*. And the force in direction A*a* to move the ſyſtem towards TR, by the power of the lever TA, is $\frac{Aa}{Ta}h\text{A} \times \text{T}a$ or $\text{A}a \times h\text{A}$. This is the centripetal force of the ſyſtem, ariſing from the gravity of A. In like manner the centripetal forces ariſing from B and C, will be B*b* × *h*B and C*c* × *h*C.

By the laws of uniform motion, P : $2\pi \times \text{A}a$: : $t : \frac{2\pi t \times \text{A}a}{\text{P}}$ = arch deſcribed by A in the time t. And $\frac{4\pi\pi tt \times \text{A}a^2}{\text{PP} \times 2\text{A}a}$ or $\frac{2\pi\pi tt \times \text{A}a}{\text{PP}}$ = diſtance it is drawn from the tangent in that time, or as the velocity generated; and therefore $\frac{2\pi\pi tt \times \text{A}a}{\text{PP}} \times \text{A}$ = motion

Fig. 5. = motion of A tending from the center *a*, by the revolution of the ſyſtem. And the force in direction *a*A, to move the ſyſtem from TR, by the power of the lever TA, will be $\frac{2\pi\pi tt \times Aa}{PP} A \times T a$. And this is the centrifugal force of the ſyſtem ariſing from the revolution of A. And in like manner the centrifugal forces ariſing from B and C, will be $\frac{2\pi\pi tt \times Bb}{PP} B \times Tb$, and $\frac{2\pi\pi tt \times Cc}{PP} C \times Tc$.

But becauſe the whole ſyſtem always keeps at the ſame diſtance from the axis TF, in its revolution; therefore the ſum of all the centripetal forces muſt be equal to the ſum of all the centrifugal forces. Whence $Aa \times hA + Bb \times hB + Cc \times hC =$ $\frac{2\pi\pi tt}{PP} \times$ into $Aa \times Ta \times A + Bb \times Tb \times B + Cc \times Tc \times C$. And conſequently $P = \pi t \sqrt{\frac{2}{h}} \times \frac{Aa \times Ta \times A + Bb \times Tb \times B + Cc \times Tc \times C}{Aa \times A + Bb \times B + Cc \times C}$.

2. If the bodies are not all in one plane, let N be the center of gravity of the bodies A, B, C. And through N draw the plane TNR; and from all the bodies, let fall perpendiculars upon that plane. Then the periodic time will be the ſame as if all the bodies were placed in theſe points where the perpendiculars cut the plane. For if *m* be one of the bodies, and *m*C perp. to the plane. Then the centripetal and centrifugal forces of *m* in direction *cm*, will be $cm \times hm$ and $\frac{2\pi\pi tt \times Tc}{PP} m \times mc$. But the force *cm* is divided into the two forces *c*C, C*m*. And all the forces C*m* deſtroy one another, becauſe the plane, TN*c*, paſſes through their center of gravity. Therefore the plane is only acted on by the

the remaining force cC. So that the centripetal and centrifugal forces will be the ſame as before, when the body was placed in C; and the periodic time is the ſame. Fig. 5.

Cor. 1. *If* Nn *be drawn from the center of gravtiy perp. to* TF; *then the periodic time of the ſyſtem,* $P = \pi t \sqrt{\frac{2}{h}} \times \frac{Ta \times Aa \times A + Tb \times Bb \times B + Tc \times Cc \times C}{Nn \times \overline{A+B+C}}$.

For (Mechan. Prop. XXXV.) $Aa \times A + Bb \times B + Cc \times C = Nn \times \overline{A+B+C}$.

Cor. 2. *The length of a ſimple pendulum, making two vibrations, or an exceeding ſmall conical motion, in the ſame periodic time, will be* $\frac{Ta \times Aa \times A + Tb \times Bb \times B + Tc \times Cc \times C}{Nn \times \overline{A+B+C}}$.

For let TO be the height of the cone deſcribed by the pendulum; then (Prop. IX) $PP = \frac{2\pi\pi tt}{h} \times TO$; therefore $TO = \frac{Ta \times Aa \times A + Tb \times Bb \times B + Tc \times Cc \times C}{Nn \times \overline{A+B+C}}$.

Cor. 3. *If* TO *be the length of an iſocronal perdulum, then* O *is the center of gravity of all the peripheries deſcribed by* A, B, C; *each multiplied by the body; whether* A, B, C *be the places of the bodies, or the points of projection upon the plane* TNR.

For if $Aa \times A$, $Bb \times B$, $Cc \times C$ be taken for bodies, their center of gravity will be diſtant from T, the length $\frac{Ta \times Aa \times A + Tb \times Bb \times B + Tc \times Cc \times C}{Aa \times A + Bb \times B + Cc \times C}$ (Mechan.

Fig. 5. chan. Prop. XXXV.) which is equal to TO by Cor. 2. and the peripheries are as the radii, A*a*, B*b*, C*c*.

Cor 4. *If any of the bodies be on the contrary ſide of the axis* TR, *or above the point of ſuſpenſion* T; *that diſtance muſt be negative.*

Cor. 5. *If any line or plane figure be placed in the plane* TNR; *then the point* O, *which gives the length of the pendulum, will be the center of gravity, of the ſurface or ſolid, deſcribed in its revolution.*

SCHOLIUM.

The point O, which gives the length of the iſocronal pendulum, is called the *center of turbination* or revolution. And the plane TNR paſſing through the center of gravity, the *turbinating plane.*

SECT. II.

The Motion of Bodies in all Sorts of CURVE LINES.

PROP. XI.

THE areas, which a revolving body deſcribes by radii drawn to a fixed center of force, are proportional to the times of deſcription; and are all in the ſame immovable plane. Fig. 6.

Let S be the center of force; and let the time be divided into very ſmall equal parts. In the firſt part of time let the body deſcribe the line AB; then if nothing hindered, it would deſcribe BK = AB, in the ſecond part of time; and then the area ASB=BSK. But in the point B let the centripetal force act by a ſingle but ſtrong impulſe, and cauſe the body to deſcribe the line BC. Draw KC parallel to SB, and compleat the parallelogram BKC*r*, then the triangle SBC = SBK, being between the ſame parallels; therefore SBC = SBA, and in the ſame plane. Alſo the body moving uniformly, would in another part of time deſcribe C*m* = CB; but at C, at the end of the ſecond part of time, let it be acted on by another impulſe, and carried along the line CD; draw *m*D parallel to CS, and D will be the place of the body after the third part of time; and the triangle SCD = SC*m* = SCB, and all in the ſame plane. After the ſame manner let the force act ſucceſſively at D, E, F, &c. And making D*n* = DC, and E*o* = ED, &c. and compleating the parallelograms as before;

Fig. 6. before; the triangle CS*m* = CSD = DS*n* = DSE = ES*o* = ESF, &c. and all in the ſame immovable plane. Therefore in equal times equal areas are deſcribed; and by compounding, the ſum of all the areas is as the time of deſcription. Now let the number of triangles be increaſed, and their breadth diminiſhed *ad infinitum*; and the centripetal force will act continually, and the figure ABCDEF, &c. will become a curve; and the areas will be proportional to the times of deſcription.

Cor. 1. *If a body deſcribes areas proportional to the times, about any point; it is urged towards that point by the centripetal force.*

For a body cannot deſcribe areas proportional to the times, about two different points or centers, in the ſame plane.

Cor. 2. *The velocity of a body revolving in a curve, is reciprocally as the perpendicular to the tangent, in that point of the curve.*

For the area of any of theſe little triangles being given; the baſe (which repreſents the velocity) is reciprocally as the perpendicular.

7. Cor. 3. *The angular velocity at the center of force, is reciprocally as the ſquare of its diſtance from that center.*

For if the ſmall triangles CSD and SBA be equal, they are deſcribed in equal times. The area $CSD = \frac{SC \times CQ}{2}$, and area $SBA = \frac{SB \times BP}{2}$, therefore $SC \times CQ = SB \times BP$. But the angle CSD : angle ASB :: CQ : *cq* :: SC × CQ : SC × *cq* :: SB × BP : SC × *cq* :: area SBA : area S*cq* :: $SB^2 : Sc^2$ or SC^2.

PROP.

Fig. 8.

PROP. XII.

If a body revolving in any curve VIL, *be urged by a centripetal force tending towards the center* S; *the centripetal force in any point* I *of the curve will be as* $\frac{\dot{p}}{p^3\dot{d}}$; *where* p = *perpendicular* SP *on the tangent at* I, *and* d = *the diſtance* SI.

For take the point K infinitely near I, and draw the lines SI, SK; and the tangents IP, K*f*; and the perpendiculars SP, S*f*. Alſo draw K*m*, K*n* parallel to SP, SI, and KN perp. to SI.

The triangles ISP, IKN, *n*K*m*, are ſimilar; as alſo IK*m*, IP*q*. Therefore I*q* or IP : IK : : *q*P : K*m*. And PS : IP : : K*m* : *mn*. And IN : IK : : *mn* : *n*K. And multiplying the terms of theſe three proportions, IP × PS × IN : IK × IP × IK : : *q*P × K*m* × *mn* : K*m* × *mn* × *n*K. That is, PS × IN : IK^2 : : *q*P : *n*K$=\frac{\text{P}q \times \text{IK}^2}{\text{PS} \times \text{IN}}$. But (Mechan. Prop. VI.) the ſpace *n*K, through which the body is drawn from the tangent, is as the force and ſquare of the time; that is (Prop. XI.) as the force and ſquare of the area ISK, or as the force × SI^2 × KN^2, or becauſe SI × KN = twice the triangle ISK = IK × SP; therefore *n*K is as the force × IK^2 × PS^2. Therefore the force at I is as $\frac{n\text{K}}{\text{IK}^2 \times \text{PS}^2} = \frac{\text{P}q \times \text{IK}^2}{\text{PS} \times \text{IN} \times \text{IK}^2 \times \text{PS}^2} = \frac{\text{P}q}{\text{PS}^3 \times \text{IN}} = \frac{\dot{p}}{p^3\dot{d}}$.

Cor. 1. *The centripetal force at* I *is as* $\frac{n\text{K}}{\text{SI}^2 \times \text{KN}^2}$, *or as* $\frac{n\text{K}}{\text{SP}^2 \times \text{IK}^2}$.

Fig. 8. Cor. 2. *Hence the radius of curvature in* I, *is* $= \frac{SI \times IN}{Pq}$.

For that radius $= \frac{IK^2}{Km} =$ (by the similar triangles IKm, IqP) $\frac{IK \times IP}{Pq} =$ (by the similar triangles IPS, INK) $\frac{SI \times IN}{Pq}$.

Cor. 3. *If* R *be the radius of curvature in* I, *the centripetal force at* I *will be as* $\frac{d}{p^3 R}$ or $\frac{SI}{R \times SP}$.

For $R = \frac{SI \times IN}{Pq}$, and $\frac{Pq}{IN} = \frac{SI}{R}$; whence the force $\frac{Pq}{PS^3 \times IN} = \frac{SI}{R \times PS^3}$.

PROP. XIII. *Prob.*

To find the law of the centripetal force, requiſite to make a body move in a given curve line.

Let the diſtance SI $= d$, the perpendicular SP (upon the tangent at I) $= p$; then from the nature of the curve, find the value of p in terms of d, and ſubſtitute it and its fluxion, in the quantity $\frac{\dot{p}}{p^3 \dot{d}}$.

Or find the value of $\frac{nK}{SI^2 \times KN^2}$ or $\frac{nK}{SP^2 \times IK^2}$ Any of theſe will give the law of centripetal force by the laſt Prop.

Ex. 1.

9. *If a body revolves in the circumference of a circle; to find the force directed to a given point* S.

Draw

Draw SI to the body at I, SP perp. to the tangent PI, SG perp. to the radius CI. Then SP = GI; because SGIP is a parallelogram. Put SI = d, SP = p, SC = a, CI = r, CD = x, ID being perp. to SD. Then in the obtuse angle SCI, $SI^2 = SC^2 + CI^2 + 2SCD$, or $dd = aa + rr + 2ax$; whence $x = \frac{dd - aa - rr}{2a}$. The triangles SCG and CID are similar, whence CI (r) : CD (x) : : SC (a) : CG = $\frac{ax}{r} = \frac{dd - aa - rr}{2r}$; and $p = r + \frac{dd - aa - rr}{2r} = \frac{dd + rr - aa}{2r}$; and $\dot{p} = \frac{d\dot{d}}{r}$. Therefore the force $\left(\frac{\dot{p}}{p^3 \dot{d}}\right)$ is as $\frac{d\dot{d}}{rp^3\dot{d}} = \frac{d}{rp^3} = \frac{d \times 8r^3}{r \times \overline{dd + rr - aa}^3}$; that is, the force is as $\frac{d}{\overline{dd + rr - aa}^3}$.

Fig. 9.

And if $a = r$, the force is as $\frac{1}{d^5}$.

Ex. 2.

If a body revolves in an ellipsis; to find the force tending to the center C. 10.

Let $\frac{1}{2}$ transverse CV = r, $\frac{1}{2}$ conjugate CD = c, draw CI = d, and its semiconjugate CR = b. Then by the properties of the ellipsis (Con. Sect. B. I. Prop. XXXIV.) $bb + dd = rr + cc$, whence $b = \sqrt{rr + cc - dd}$; and (ib. Prop. XXXVII.) b or $\sqrt{rr + cc - dd} : c : : r : p = \frac{cr}{\sqrt{rr + cc - dd}}$; and $\dot{p} = \frac{crd\dot{d}}{\overline{rr + cc - dd}^{\frac{3}{2}}}$. There-

Fig. 10. fore $\frac{\dot{p}}{p^3\dot{d}} = \frac{crd\dot{d}}{\overline{rr+cc-dd}^{\frac{3}{2}}} \times \frac{\overline{rr+cc-dd}^{\frac{3}{2}}}{c^3r^3\dot{d}} = \frac{\dot{d}}{ccrr}$.

Therefore the force is directly as the distance CI.

After the same manner, the force tending to the center of an hyperbola, will be found $\frac{-d}{ccrr}$, which is a centrifugal force, directly as the distance.

Ex. 3.

11. *If a body revolves in an ellipsis, to find the law of centripetal force, tending to the focus* S.

Let the semitransverse OV $= r$, the semiconjugate OD $= c$, draw SI $= d$; and OI, and its conjugate OK $= b$.

Then (Con. Sect. B. I. Prop. XXXV.) $2rd-dd = bb$; and (ib. Prop. XXXVI.) b or $\sqrt{2rd-dd}$: $c :: d : p = \frac{cd}{\sqrt{2rd-dd}}$; and $\dot{p} =$

$$\frac{c\dot{d}\sqrt{2rd-dd} - cd \times \overline{2rd-dd}^{-\frac{1}{2}} \times \overline{r\dot{d}-d\dot{d}}}{2rd-dd} =$$

$$\frac{c\dot{d} \times \overline{2rd-dd} - cd \times \overline{r\dot{d}-d\dot{d}}}{\overline{2rd-dd}^{\frac{3}{2}}} = \frac{crd\dot{d}}{\overline{2rd-dd}^{\frac{3}{2}}}.$$

Therefore $\frac{\dot{p}}{p^3\dot{d}} = \frac{crd\dot{d} \times \overline{2dr-dd}^{\frac{3}{2}}}{c^3d^3\dot{d} \times \overline{2dr-dd}^{\frac{3}{2}}} = \frac{crd}{c^3d^3} = \frac{r}{ccdd}$.

Therefore the centripetal force is as $\frac{1}{dd}$, or reciprocally as the square of the distance.

Ex. 4.

12. *If a body revolves in the hyperbola* VI; *to find the law of centripetal force, tending to the focus* S.

Draw SI, and the tangent IT, and SP perp. upon it. And let the semitransverse SO $= r$, semiconjugate

miconjugate $= c$, SI $= d$, SP $= p$, and $b =$ semi- Fig. conjugate to IO. 12.

Then (Con. Sect. B. II. Prop. XXXI.) $2rd + dd = bb$, and $b = \sqrt{2rd + dd}$. And (ib. Prop. XXXII.) b or $\sqrt{2rd + dd} : c :: d : p = \frac{cd}{\sqrt{2rd+dd}}$; whence $\dot{p} =$

$$\frac{\dot{cd} \times \sqrt{2rd + dd} - cd \times \overline{2rd + dd}^{-\frac{1}{2}} \times \overline{r\dot{d} + d\dot{d}}}{2rd + dd}$$

$$= \frac{\dot{cd} \times \overline{2rd + dd} - cd \times \overline{r\dot{d} + d\dot{d}}}{\overline{2rd + dd}^{\frac{3}{2}}} = \frac{crd\dot{d}}{\overline{2rd + dd}^{\frac{3}{2}}}.$$

Therefore $\frac{\dot{p}}{p^3\dot{d}} = \frac{crd\dot{d} \times \overline{2rd + dd}^{\frac{3}{2}}}{\overline{2rd + dd}^{\frac{3}{2}}\dot{d} \times c^3d^3} = \frac{crd}{c^3d^3} = \frac{r}{ccdd}$. Therefore the centripetal force is as $\frac{r}{ccdd}$ or $\frac{1}{dd}$; that is, reciprocally as the square of the distance.

And in like manner the force towards the other focus F, is $\frac{-r}{ccdd}$, or as $\frac{-1}{dd}$, which is a centrifugal force reciprocally as the square of the distance.

Ex. 5.

If a body revolves in the parabola VI; *to find the force tending to the focus* S. 13.

Draw IS, and the tangent IT, and SP perp. to it. And put SI $= d$, SP $= p$, *latus rectum* $= r$. Then (Con. Sect. B. III. Prop. II. and Cor. 3. Prop. XII.) $pp = \frac{1}{4}rd$, and $2p\dot{p} = \frac{1}{4}rd\dot{}$; and $\dot{p} = \frac{r\dot{d}}{8p} = \frac{rd\dot{}}{4\sqrt{rd}}$. And $\frac{\dot{p}}{p^3\dot{d}} = \frac{rd\dot{}}{4\sqrt{rd} \times \frac{1}{8}rd\sqrt{rd} \times \dot{d}} =$

Fig. 13. $= \frac{8r}{4rrdd} = \frac{2}{rdd}$. Therefore the centripetal force is reciprocally as the ſquare of the diſtance CI.

Hence, in all the Conic Sections the centripetal force tending to the focus, is reciprocally as the ſquare of the diſtance from the focus.

Ex. 6.

14. *Let* VI *be the logarithmic ſpiral, to find the force tending to the center* S.

Draw the tangent IP, and SP perp. to it, let SI $= d$, SP $= p$; then the ratio of d to p is always given, ſuppoſe as m to n. Then $p = \frac{n}{m}d$, and $\dot{p} = \frac{n}{m}\dot{d}$. Conſequently $\frac{\dot{p}}{p^3\dot{d}} = \frac{n\dot{d}}{m} \times \frac{m^3}{n^3d^3\dot{d}} = \frac{mm}{nnd^3}$; and the centripetal force is as $\frac{1}{d^3}$, or reciprocally as the cube of the diſtance.

PROP. XIV.

15. *The velocity of a body moving in any curve* QAO, *in any point* A; *is to the velocity of a body moving in a circle at the ſame diſtance; as* $\sqrt{\dot{p}d}$ *to* $\sqrt{\dot{d}p}$. *Putting* d *=diſtance* SA, *and* p *=*SP *the perpendicular on the tangent at* A.

Let AR be the radius of curvature; from the point a in the curve infinitely near A, draw am, an parallel to AS, AR. Let C=velocity in the curve; c=velocity in the circle. By ſimilar triangle SP (p) : SA (d) : : an : am : : centripetal force tending to R : centripetal force tending to S : ; (Prop. V. Cor. 2.) $\frac{CC}{AR} : \frac{cc}{AS}$. But (Prop. XII. Cor. 2.) $AR = \frac{d\dot{d}}{\dot{p}}$; whence $p : d : : \frac{CC\dot{p}}{\dot{d}d} :$

Fig. 15.

$\frac{cc}{d} :: CC\dot{p} : cc\dot{d}$. And $p\dot{d} : d\dot{p} :: CC : cc$.

Cor. 1. *If* r *= half the tranverſe axis of an ellipſis; then the velocity of a body revolving round the focus is to that in a circle at the ſame diſtance; as* $\sqrt{2r-d} : \sqrt{r}$.

For $\dot{p} = \frac{cr\dot{d}\dot{d}}{\overline{2rd-dd}^{\frac{3}{2}}}$ (See Ex. 3. Prop. XIII.), and $p = \frac{cd}{\sqrt{2rd-dd}}$. And the ſquares of the velocities in the curve, and in the circle, are as $\frac{cd\dot{d}}{\sqrt{2rd-dd}}$ and $\frac{cr\dot{d}\dot{d}}{\overline{2rd-dd}^{\frac{3}{2}}}$, or as 1 and $\frac{rd}{2rd-dd}$, or as $2r-d$ to r.

Cor. 2. *Suppoſe as before, the velocity of a body revolving round the center of an ellipſis, is to the velocity in a circle at the ſame diſtance; as half the conjugate diameter to that diſtance, is to the diſtance.*

For $p = \frac{cr}{\sqrt{rr+cc-dd}}$, and $\dot{p} = \frac{cr\dot{d}\dot{d}}{\overline{rr+cc-dd}^{\frac{3}{2}}}$. Whence, the ſquares of theſe velocities are as $\frac{cr\dot{d}}{\sqrt{rr+cc-dd}}$ and $\frac{cr\dot{d}\dot{d}\dot{d}}{\overline{rr+cc-dd}^{\frac{3}{2}}}$ or as 1 to $\frac{dd}{rr+cc-dd}$ or as $rr+cc-dd$ to dd, or as bb to dd. See Ex. 2. Prop. XIII.

Cor. 3. *The velocity in a parabola round the focus, is to the velocity in a circle at the ſame diſtance; as* $\sqrt{2}$ *to* 1.

For $p = \frac{1}{2}\sqrt{rd}$, and $\dot{p} = \frac{r\dot{d}}{4\sqrt{rd}}$ (See Ex. 5.

Fig. 15. Prop. XIII.) Whence the ſquares of theſe velocities are as $\frac{1}{2}\dot{d}\sqrt{rd}$ and $\frac{rd\dot{d}}{4\sqrt{rd}}$, or as $\frac{1}{2}rd$ to $\frac{1}{4}rd$; that is as 2 to 1.

Cor. 4. *The velocity of a body in the logarithmic ſpiral in any point, is the ſame as the velocity of a body at the ſame diſtance in a circle.*

For $p=\frac{n}{m}d$, and $\dot{p}=\frac{n}{m}\dot{d}$, (Ex. 6. Prop. XIII.)

And the ſquares of the velocities are as $\frac{n}{m}d\dot{d}$ and $\frac{n}{m}d\dot{d}$, that is, equal.

PROP. XV. *Prob.*

16. *To find the force which, acting in direction of the ordinate* MP, *ſhall cauſe the body to move in that curve.*

Draw *mp* parallel and infinitely near MP, and M*s* parallel to AP. Then univerſally the force is always as the ſpace *mr*, through which it is drawn from the tangent, in a given time. But *ms* or *rs* is the fluxion and *mr* the ſecond fluxion of the ordinate PM. Therefore making the fluxion of the time conſtant; or which is the ſame thing, making the fluxion of the axis conſtant; find the ſecond fluxion of the ordinate, which will be as the force.

Ex. 1.

Let the curve be an ellipſis whoſe equation is $y=\frac{c}{r}\sqrt{2rx-xx}$. Putting AP $=x$, PM $=y$, $r=$ ſemitranſverſe, $c=$ ſemiconjugate. Then $\dot{y}=\frac{c}{r}\times\frac{r\dot{x}-x\dot{x}}{\sqrt{2rx-xx}}$, and $\ddot{y}=\frac{c}{r}\times\frac{-\dot{x}\sqrt{2rx-xx}}{2rx-xx}$ =

Fig. 16.

$$-\frac{c}{r}\times\overline{r-x}\times\overline{r\dot{x}-x\dot{x}}\times\overline{2rx-xx}^{-\frac{1}{2}}=\frac{c}{r}\times\frac{-2rx+xx-r-x^2}{\overline{2rx-xx}^{\frac{3}{2}}}=\frac{c}{r}\times\frac{-rr}{\overline{2rx-xx}^{\frac{3}{2}}}=\frac{-cr}{y^3}.$$

That is, the force is as $\frac{-1}{y^3}$, or reciprocally as the cube of the ordinate. The ſame is true of the circle, which is one ſort of ellipſis.

Ex. 2.

Let the curve be a parabola, AP $= x$, PM $= y$, and $r\dot{x}=y\dot{y}$; then $r\dot{\dot{x}}=2\dot{y}\dot{y}$, and $2\dot{y}\dot{\dot{y}}+2y\ddot{y}=0$; therefore $y\ddot{y}=-\dot{y}\dot{y}$, and $\ddot{y}=-\frac{\dot{y}\dot{y}}{y}=-\frac{r\dot{\dot{x}}\dot{x}}{4yy\times y}=\frac{-rr}{4y^3}$, and the force as $\frac{-1}{y^3}$, or reciprocally as the cube of the ordinate.

PROP. XVI.

If the law of centripetal force be reciprocally as the ſquare of the diſtance; the velocities of bodies revolving in different ellipſes about one common center, are directly as the ſquare roots of the parameters, and reciprocally as the perpendiculars to the tangents at theſe points of their orbits.

Let d, D be the diſtances in two ellipſes; r, c, l, p; and R, C, L, P, the ſemitranſverſe, ſemiconjugate, *latus rectum*, and perpendicular in the two ellipſes. Then the ſquares of the velocities in two circles whoſe radii are d, D, (by Prop. IV. Cor. 2.) will be as $d\times$ force in d, and D $\times$ force in D; that is, as $\frac{d}{dd}$ and $\frac{D}{DD}$ or as $\frac{1}{d}$ and $\frac{1}{D}$.

Then (Prop. XIV. Cor. 1.),

velocity

Fig. velocity in the ellipsis d : vel. in the circle d : : $\sqrt{2r-d}$: $\sqrt{r}$.
and vel. in the circle d : vel. in the circle D : : $\sqrt{\frac{1}{d}}$: $\sqrt{\frac{1}{D}}$.

And (Prop. XIV. Cor. 1.)
vel. in the circle D : vel. in the ellipsis D : : $\sqrt{R}$: $\sqrt{2R-D}$.
Therefore vel. in the ellipsis d : vel. in the ellipsis D : : $\sqrt{\frac{2r-d \times R}{d}}$: $\sqrt{\frac{2R-D \times r}{D}}$: : $\sqrt{\frac{2r-d}{dr}}$: $\sqrt{\frac{2R-D}{DR}}$.

But (Con. Sect. B. I. Prop. 36.) $p = c\sqrt{\frac{d}{2r-d}}$, and $p\sqrt{\frac{2r-d}{d}} = c$, and $\sqrt{\frac{2r-d}{d}} = \frac{c}{p}$, and $\sqrt{\frac{2r-d}{dr}} = \frac{c}{p\sqrt{r}} = \frac{\sqrt{\frac{1}{2}l}}{p}$ (becauſe $\frac{cc}{r} = \frac{1}{2}l$ by the Conic Sections.) In like manner $\sqrt{\frac{2R-D}{DR}} = \frac{\sqrt{\frac{1}{2}L}}{P}$. Whence, vel. in the ellipsis d : vel. in the ellipsis D : : $\frac{\sqrt{l}}{p}$: $\frac{\sqrt{L}}{P}$.

Cor. 1. *Hence the velocities in the two ellipſis, are as* $\sqrt{\frac{2r-d}{dr}}$, *and* $\sqrt{\frac{2R-D}{DR}}$.

Cor. 2. *Alſo the ſquares of the areas deſcribed in the ſame time, are as the parameters.*

For the areas are as the arches × perpendiculars, or as the velocities × perpendiculars; that is, as $\frac{\sqrt{l}}{p} \times p$ and $\frac{\sqrt{L}}{P} \times P$, or as $\sqrt{l}$ and $\sqrt{L}$.

Cor. 3. *The velocity of a body in different parts of its orbit is reciprocally as the perpendicular upon the tangent*

tangent at that point; and therefore is as $\sqrt{\frac{2r-d}{d}}$. Fig.

For the parameter is given.

Cor. 4. *The velocity in a conic ſection at its greateſt or leaſt diſtance, is to the velocity in a circle at the ſame diſtance; as the ſquare root of the parameter, to the ſquare root of twice that diſtance.*

For here $d=D=p=P$, and $L=2D$. Therefore the velocity in the ellipſis, to the velocity in the circle; as $\frac{\sqrt{l}}{P} : \frac{\sqrt{2D}}{P} :: \sqrt{l} : \sqrt{2D}$.

Cor. 5. *The velocity in an ellipſis at its mean diſtance, is the ſame as in a circle at the ſame diſtance.*

For if d be the mean diſtance, then $p=c$. And if D be the radius of the circle, then $L=2D$, and $P=D$. Whence, vel. in the ellipſis : to the vel. in the circle $:: \frac{\sqrt{l}}{c} : \frac{\sqrt{2D}}{D} ::$ (becauſe $cc=\frac{1}{2}lr$)

$\frac{1}{\sqrt{\frac{1}{2}r}} : \frac{1}{\sqrt{\frac{1}{2}D}} :: \sqrt{D} : \sqrt{r}$. But $D=r$, therefore the velocites are equal.

Cor. 6. *Both the real and apparent velocity round the focus* F, *is greateſt at* A, *the neareſt vertex; and leaſt at* B, *the remote vertex.* 17.

For the real velocity is reciprocally as the perpendicular, which is leaſt at A and greateſt at B. And the apparent velocity at F is reciprocally as the ſquare of the diſtance from F, which diſtance is leaſt at A, and greateſt at B, (Cor. 2. and 3, Prop. XI.)

Cor. 7. *The ſame things ſuppoſed, and* PC, CK *being ſemiconjugates; the velocity in the curve, is to the velocity towards the focus* F; *as* CK *to* $\sqrt{CK^2-CD^2}$. 23.

For

Fig. 23. For vel. in the curve : vel. towards F : : Pp : pn : : FP : NP : : d : $\sqrt{dd-pp}$. But $pp = \frac{ccd}{2r-d}$, and $dd-pp = \frac{2rd-dd-cc}{2r-d}d$. Whence vel. in the curve : vel. towards F : : d ; $\sqrt{\frac{2rd-dd-cc}{2r-d}}d$: : $\sqrt{2rd-dd}$: $\sqrt{2rd-dd-cc}$: : (Con. Sect. B. I. Prop. XXXV.) CK : $\sqrt{CK^2-CD^2}$.

Cor. 8. *The aſcending or deſcending velocity is the greateſt when* FP *is half the* latus rectum, *or when* FP *is perp. to* AB.

For $\sqrt{2rd-dd}$: $\sqrt{2rd-dd-cc}$: : vel. in the curve $\left(\frac{\sqrt{2r-d}}{d}\right)$: vel. towards F = $\sqrt{\frac{2rd-dd-cc}{dd}}$; and making the ſquare of this velocity a maximum, then $\frac{2rd-dd-cc}{dd} = m$, and $\overline{2rd-2dd} \times dd - 2dd \times \overline{2rd-dd-cc} = 0$; and $rd-dd-2rd+dd+cc = 0$, and $-rd+cc = 0$. Whence $d = \frac{cc}{r}$ = half the *latus rectum*.

Cor. 9. *If* FR, *the diſtance from the focus to the curve be* $= \sqrt{CA \times CD}$; *then* R *is the place where the angular motion about the focus* F, *is equal to the mean motion.*

For the area of a circle whoſe radius FR is $= \sqrt{CA \times CD}$ is equal to the area of the ellipſis; and if we ſuppoſe them both deſcribed in equal times, then the ſmall equal parts at R will be deſcribed in equal times; and therefore the angular velocities at F will be equal; and both equal to the mean motion. The angular motion in the ellipſis

lipſis from B to R will be ſlower; and from R to A ſwifter, than the mean motion. Fig.

PROP. XVII.

If the centripetal forces be reciprocally as the ſquares of the diſtances; the periodic times in ellipſes, will be in the ſeſquiplicate ratio of the tranſverſe axes AB; *or the ſquares of the periodic times, will be as the cubes of the mean diſtances* FD, *from the common center.* 17.

Put the ſimbols as in the laſt, and t, T, for the periodical times. Then by the nature of the ellipſis $cc = \frac{1}{2}lr$, and $c = \sqrt{\frac{1}{2}lr}$, and $rc = r\sqrt{\frac{1}{2}lr}$. And for the ſame reaſon $RC = R\sqrt{\frac{1}{2}LR}$. Alſo (Prop. XVI. Cor. 2.) the areas deſcribed in the ſame time are as the ſquare roots of the parameters; and therefore the whole areas of the ellipſes, are as the periodical times multiplied by the ſquare roots of the parameters. But the whole areas are alſo as the rectangles of the axes; therefore the rectangles of the axes are as the periodical times multiplied by the ſquare roots of the parameters; that is, rc or $r\sqrt{\frac{1}{2}lr} : RC$ or $R\sqrt{\frac{1}{2}LR} :: t\sqrt{l} : T\sqrt{L}$. And ſquaring, $\frac{1}{2}lr^3 : \frac{1}{2}LR^3 :: ttl : TTL$. That is, $r^3 : R^3 :: tt : TT$. And $t : T :: r^{\frac{3}{2}} : R^{\frac{3}{2}} :: \overline{2r}^{\frac{3}{2}} : \overline{2R}^{\frac{3}{2}}$.

Cor. 1. *The areas of the ellipſes are as the periodic times multiplied by the ſquare roots of the parameters.*

Cor. 2. *The periodic time in an ellipſis, is the ſame as in a circle, whoſe diameter is equal to the tranſverſe axis* AB; *or the radius equal to the mean diſtance* ED.

Cor. 3. *The quantities of matter in central attracting bodies, that have others revolving about them in ellipſes; are as the cubes of the mean diſtances, divided by the ſquares of the periodical times.*

For

Fig. 17. For (Cor. 2.) the periodic times are the ſame when the mean diſtances are equal to the radii; and the reſt follows from Prop. VII.

Cor. 4. *Whatever has been demonſtrated in Sect. I. concerning the forces, velocities and periodic times of bodies moving in circles; hold equally true in ellipſes, taking the mean diſtances inſtead of the radii.*

This will appear by Cor. 2. or by comparing this Prop. with Cor. 5. Prop. V. for the periodic times. And by Cor. 5. Prop. XVI. for the velocities; the forces being reciprocally as the ſquares of the diſtances, in both caſes.

PROP. XVIII.

18. *If the centripetal forces be directly as the diſtances; the periodic times of bodies moving in ellipſes round the ſame center, will be all equal to one another.*

Let AEL be an ellipſis, AGL a circle on the ſame axis AL, C the center of both. Draw the tangent AD, and *np*F parallel to it, and D*n*, B*p* parallel to AC : AF being very ſmall. Then *Dn* equal to B*p* will be as the centripetal force; and therefore AD and AB, or A*n* and A*p* will be deſcribed in the ſame time, in the circle and ellipſis. Conſequently the areas deſcribed in theſe equal times will be A*n*C and A*p*C. But theſe areas are to one another as *n*F to PE, or as GC to EC; that is, as the area of the circle AGL to the area of the ellipſis AEL. Therefore ſince parts proportional to the wholes are deſcribed in equal times; the wholes will be deſcribed in equal times. And therefore the periodic times, in the circle and ellipſis, are equal.

But (Prop. V. Cor. 4.) the periodic times in all circles are equal, in this law of centripetal force; and therefore the periodic times in all ellipſes are equal.

Cor.

Cor. *The velocity at any point* I *of an ellipſis, is as the rectangle of the two axes* AC, CE; *divided by the perpendicular* CH, *upon the tangent at* I. Fig. 18.

For the arch I × CH is as the area deſcribed in a ſmall given part of time, and that is as the whole area (becauſe the periodic times are equal) or as AC × CE. And therefore the arch I or the velocity, is as $\frac{AC \times CE}{CH}$.

PROP. XIX.

The denſities of central attracting bodies, are reciprocally as the cubes of the parallaxes of the bodies revolving about them (as ſeen from theſe central bodies) and reciprocally as the ſquares of the periodic times.

For the denſity multiplied by the cube of the diameter, is as the quantity of matter; that is (by Prop. XVII. Cor. 3.) as the cube of the mean diſtance divided by the ſquare of the periodical time of the revolving body. And therefore the denſity is as the cube of the diſtance, divided by the cube of the diameter, and by the ſquare of the periodic time. But the diameter divided by the diſtance is as the angle of the parallax; therefore the denſity is as 1 divided by the cube of the parallax, and the ſquare of the periodic time.

Cor. *The denſities of central attracting bodies (having others revolving about them) are reciprocally as the cubes of the apparent diameters of theſe central bodies (ſeen from the revolviug bodies), and reciprocally as the ſquares of the periodic times.*

PROP. XX.

If two bodies A, B, *revolve about each other; they will both of them revolve about their center of gravity.* 19.

Let

Fig. 19. Let C be the center of gravity of the bodies A, B, acting upon one another by any centripetal forces. And let AZ be the direction of A's motion; draw BM parallel to AZ, for the direction of B. And let AZ, BH be deſcribed in a very ſmall part of time, ſo that AZ may be to BH, as AC to BC; and then C will be the center of gravity of Z and H, becauſe the triangles ACZ and BCH are ſimilar. Whence AC : CB : : ZC : CH. But as the bodies A and B attract one another, the ſpaces *Aa* and *Bb* they are drawn through will be reciprocally as the bodies, or directly as the diſtances from the center of gravity; that is *Aa* : *Bb* : : AC : BC. Compleat the parallelograms *Ac* and *Bd*; and the bodies, inſtead of being at Z and H, will be at *c* and *d*. But ſince AC : BC : : *Aa* : *Bb*. By diviſion AC : BC : : *a*C : *b*C. But AC : BC : : AZ : BH : : *ac* : *bd*. Whence *a*C : *b*C : : *ac* : *bd*. Therefore the triangles *c*C*a*, and *d*C*b* are ſimilar, whence C*c* : C*d* : : *ac* : *bd* : : AC : BC : : B : A. Therefore C is ſtill the center of gravity of the bodies at *c* and *d*.

In like manner, producing B*d* and A*c*, till *dg* be equal to B*d*, and *cq* to A*c*; and if *cf*, *dh*, be the ſpaces drawn through by their mutual attractions; and if the parallelograms *ce*, *di*, be compleated. Then it will be proved by the ſame way of reaſoning, that C is the center of gravity of the bodies at *q* and *g*, and alſo at *e* and *i*, where A deſcribes the diagonals A*c*, *ce*, &c. and B the diagonals B*d*, *di*, and ſo on *ad infinitum*.

If one of the bodies B is at reſt whilſt the other moves along the line AL. Then the center of gravity C will move uniformly along the line CO parallel to AL. Therefore if the ſpace the bodies move in, be ſuppoſed to move in direction CO, with the velocity of the center of gravity; then the center of gravity will be at reſt in that ſpace, and

and the body B will move in direction BH parallel to CO or AZ; and then this case comes to the same as the former. Therefore the bodies will always move round the center of gravity, which is either at rest, or moves uniformly in a right line. Fig. 19.

If the bodies repel one another; by a like reasoning it may be proved that they will constantly move round their center of gravity.

If the lines CA, C*c*, C*e*, &c. be equal; and CB, C*d*, C*i*, &c. also equal. Then it is the case of two bodies joined by a rod or string; or of one body composed of two parts. This body or bodies will always move round their common center of gravity.

Cor. 1. *The directions of the bodies in opposite points of the orbits, are always parallel to one another.*

For since AZ : Z*c* : : BH : H*d*; and AZ, Z*c* parallel to BH, H*d*; therefore the $\angle$ ZA*c* = $\angle$ HB*d*, and B*d* parallel to A*c*. And for the same reason *di* is parallel to *ce*, &c.

Cor. 2. *Two bodies, acting upon one another by any forces; describe similar figures about their common center of gravity.*

For the particles A*c*, B*d* of the curves are parallel to one another, and every where proportional to the distances of the bodies AC, BC.

Cor. 3. *If the forces be directly as the distances; the bodies will describe concentrical ellipses round the center of gravity.*

Cor. 4. *If the forces be reciprocally as the squares of the distances; the bodies will describe similar ellipses or some conic sections, about each other, whose center of gravity is in the focus of both.*

PROP. XXI.

Fig. 20. *If two bodies* S, P, *attract each other with any forces, and at the ſame time revolve about their center of gravity* C. *Then if either body* P, *with the ſame force, deſcribes a ſimilar curve about the other body* S *at reſt; its periodical time, will be to the periodical time of either about the center of gravity; as the ſquare root of the ſum of the bodies* ($\sqrt{S+P}$), *to the ſquare root of the fixed or central body* ($\sqrt{S}$.)

Let PV be the orbit deſcribed about C, and P*v* that deſcribed about S. Draw the tangent P*r*, take the arch PQ extremely ſmall, and draw CQR, alſo draw S*qr* parallel to CR, and then PQ and P*q* will be ſimilar parts of the curves PV and P*v*.

Now the times that the bodies are drawn from the tangent through the ſpaces QR, *qr*, with the ſame force, will be as the ſquare roots of the ſpaces QR, *qr*; that is (becauſe of the ſimilar figures CPRQ and SP*rq*) as $\sqrt{CP}$ to $\sqrt{SP}$; that is, (by the nature of the center of gravity) as $\sqrt{S}$ to $\sqrt{S+P}$. But the times wherein the bodies are drawn from the tangent through RQ, *rq*, are the times wherein the ſimilar arches PQ, P*q* are deſcribed; and theſe times are as the whole periodic times. Therefore the periodic time in PV, is to the periodic time in P*v*; as $\sqrt{S}$ to $\sqrt{S+P}$.

Cor. 1. *The velocity in the orbit* PV *about* C, *is to the velocity in the orbit* P*v about* S; *as* $\sqrt{S}$ *to* $\sqrt{S+P}$.

For the velocities are as the ſpaces divided by the times; therefore, vel. in PV : vel. in P*v* : : $\frac{PQ}{\sqrt{S}} : \frac{Pq}{\sqrt{S+P}} :: \frac{CP}{\sqrt{S}} : \frac{SP}{\sqrt{S+P}} :: \frac{S}{\sqrt{S}} : \frac{S+P}{\sqrt{S+P}}$: : $\sqrt{S} : \sqrt{S+P}$.

Cor.

Cor. 2. *Bodies revolving round their common center of gravity, deſcribe areas proportional to the times.* Fig. 20.

PROP. XXII.

If the forces be reciprocally as the ſquares of the diſtances; and if a body revolves about the center L *in the ſame periodical time, that the bodies* S, P, *revolve about the center of gravity* C. *Then will* SP : LP : : $\sqrt[3]{S+P} : \sqrt[3]{S}$. 20.

Let PN be the orbit deſcribed about L. Then (Prop. XXI.) per. time in PQ : per. time in P*q* : : $\sqrt{S} : \sqrt{S+P}$: : $\sqrt{CP} : \sqrt{SP}$. And (Prop. XVII.) per. time in P*q* : per. time in PN : : $SP^{\frac{3}{2}} : LP^{\frac{3}{2}}$; ſuppoſing PQ, PN, ſimilar arches, and the body S placed in L. Therefore per. time in PQ : per. time in PN : : $\sqrt{CP \times SP^{\frac{3}{2}}} : \sqrt{SP \times LP^{\frac{3}{2}}}$: : $\sqrt{CP \times SP^2} : \sqrt{LP^3}$. But the periodic times are equal; therefore $\sqrt{CP \times SP^2} = \sqrt{LP^3}$, and $LP^3 = CP \times SP^2$, and $LP = \sqrt[3]{CP \times SP^2}$. But $LP : SP : : \sqrt[3]{CP \times SP^2} : SP$ or $\sqrt[3]{SP^3}$: : $\sqrt[3]{CP} : \sqrt[3]{SP}$: : $\sqrt[3]{S} : \sqrt[3]{S+P}$.

Cor. 1. *If the forces be reciprocally as the ſquare of the diſtances; the mean diſtance* (SP) *of the bodies moving about their center of gravity* C; *is to their mean diſtance when either body* P *moves about the other body* S *at reſt, in the ſame periodic time; as the cube root of the ſum of the bodies* S + P, *to the cube root of the fixt or central body* S.

For in fig. 20. make SP = LP, the mean diſtance; then SP : S*p* : : $\sqrt[3]{S+P} : \sqrt[3]{S}$.

Cor. 2. *If the forces be reciprocally as the ſquares of the diſtances; the tranſverſe axis of the ellipſis*

Fig. 20. *deſcribed by either body* P *about the center of gravity* C, *is to the tranſverſe axis deſcribed by* P *about the other body* S *at reſt, in the ſame periodical time*; *as the cube root of the ſquare of the central body* ($\sqrt[3]{SS}$) *to the cube root of the ſquare of the ſum of the bodies* ($\sqrt[3]{\overline{S+P}^2}$)

For $Sp : SP :: \sqrt[3]{CP} : \sqrt[3]{SP}$.

$SP : CP :: \sqrt[3]{SP^3} : \sqrt[3]{CP^3}$, therefore it is $Sp : CP :: \sqrt[3]{CP \times SP^3} : \sqrt[3]{CP^3 \times SP} :: \sqrt[3]{SP^2} : \sqrt[3]{CP^2} :: \sqrt[3]{\overline{S+P}^2} : \sqrt[3]{SS}$.

Or thus.

$$PL : PC :: PS^2 : PL^2 :: \overline{S+P}^{\frac{2}{3}} : S^{\frac{2}{3}}.$$

Cor. 3. *If two bodies attracting each other move about their center of gravity. Their motions will be the ſame as if they did not attract one another, but were both attracted with the ſame forces, by another body placed in the center of gravity.*

PROP. XXIII. *Prob.*

21. *Suppoſe the centripetal force to be directly as the diſtance. To determine the orbit which a body will deſcribe, that is projected from a given place* P, *with a given velocity, in a given direction* PT.

By Ex. 2. Prop. XIII. the body will move in an ellipſis, whoſe center is C the center of force; and the line of direction PT will be a tangent at the point P. Draw CR perp. to PT. And let the diſtance $CP = d$. $CR = p$, ſemitranſverſe axis $CA = R$, ſemiconjugate axis $CB = C$. CG (the ſemiconjugate to CP) $= B$. $f =$ ſpace a body would deſcend at P, in a ſecond, by the centripetal force. $v =$ the velocity at P, the body is projected with, or the ſpace it deſcribes in a ſecond.

Then

Then $\sqrt{2df}$ = velocity of a body revolving in a circle at the diſtance CP. Fig. 21.

Then (Prop. XIV. Cor. 2.) $v : \sqrt{2df} :: B : d$, and $B\sqrt{2df} = dv$, and $2BBdf = ddvv$, whence $BB = \frac{dvv}{2f}$, and $B = v\sqrt{\frac{d}{2f}}$. But (Con. Sect. B. I. Prop. XXXIV.) $RR + CC = BB + dd = \frac{vvd}{2f} + dd$. And (ib. Prop. XXXVII.) $CR = Bp = pv\sqrt{\frac{d}{2f}}$. Therefore $RR + CC + 2RC = \frac{vvd}{2f} + dd + 2pv\sqrt{\frac{d}{2f}}$, and $R + C = \sqrt{\frac{vvd}{2f} + dd + 2pv\sqrt{\frac{d}{2f}}} = m$. Alſo $RR + CC - 2RC = \frac{vvd}{2f} + dd - 2pv\sqrt{\frac{d}{2f}}$, and $R - C = \sqrt{\frac{vvd}{2f} + dd - 2pv\sqrt{\frac{d}{2f}}} = n$. Therefore $R = \frac{m+n}{2}$, and $C = \frac{m-n}{2}$.

Then to find the poſition of the tranſverſe axis AD. Let F, S be the foci. Then (by Con. Sect. B. I. Prop. II. Cor.) we ſhall have SC or $CF = \sqrt{RR - CC}$. Put $FP = x$; then $SP = 2R - x$, and (ib. Prop. XXXV.) $SP \times PF$ or $2Rx - xx = BB$, and $RR - 2Rx + xx = RR - BB$, and $R - x = \pm\sqrt{RR - BB}$; whence $x = R \pm \sqrt{RR - BB}$; that is, the greater part $FP = R + \sqrt{RR - BB}$, and the leſſer part $SP = R - \sqrt{RR - BB}$. Then in the triangle PCF or PCS, all the ſides are given, to find the angle PCF or PCA.

Fig. 21. Cor. *The periodical time in ſeconds, is* 3.1416 $\sqrt{\frac{2d}{f}}$.

For arch $\sqrt{2df}$: time 1″ : : circumference 3.1416 $\times 2d$: 3.1416 $\sqrt{\frac{2d}{f}}$ the periodical time in a circle whoſe radius is d. And by Prop. XVIII. the periodical time is the ſame in all circles and ellipſes.

PROP. XXIV. *Prob.*

22. *Suppoſing the centripetal force reciprocally as the ſquare of the diſtance; to determine the orbit which a body will deſcribe; that is projected from a given place* P, *with a given velocity, in a given direction* PT.

By Prop. XIII. the body will move in a conic ſection, whoſe focus is S the center of force. And the line of direction PT will be a tangent at the point P. Let the diſtance SP = d, tranſverſe axis AD = z, f = ſpace a body will deſcend at P in a ſecond, by the centripetal force, v = the velocity the body is projected with from P, or the ſpace it deſcribes in a ſecond. Then $\sqrt{2df}$ is the velocity of a body revolving in a circle at the diſtance SP.

Then (Prop. XIV. Cor. 1.) $v : \sqrt{2df} : : \sqrt{z-d} : \sqrt{\frac{1}{2}z}$. Whence $v\sqrt{\frac{1}{2}z} = \sqrt{2dfz - 2ddf}$, and $vvz = 4dfz - 4ddf$; and $4dfz - vvz = 4ddf$, whence $z = \frac{4ddf}{4df - vv}$ = AD. And PH = $z - d = \frac{dvv}{4df - vv}$. Therefore if $4df$ is greater than vv, z is affirmative, and the orbit is an ellipſis. But if leſſer, z is negative, and the curve is a hyperbola, and if equal, it is a parabola.

Draw SR perp. to PT, and let SR = p. Alſo draw from the other focus H, HF perp. to PT.

Then

Then (Con. Sect. B. I. Prop. X.) the angle SPR = angle HPF, whence the triangles SPR, HPF are similar; therefore SP (d) : SR (p) : : HP ($z-d$) : HF $= \frac{z-d}{d}p$; and (ib. Prop. XXI.) SR × HF or $\frac{z-d}{d}pp$ = rectangle DHA or CB^2, the square of half the conjugate axis; therefore $CB = p\sqrt{\frac{z-d}{d}}$. Fig. 22.

In the triangle SPH; the angle SPH and the sides SP, PH are given, to find the angle PSH, the position of the transverse axis.

Cor. 1. *The periodical time in the ellipsis* APDB $= 3.1416 \times \frac{4ddf}{\overline{4df-vv}^{\frac{3}{2}}}$.

For $3.1416\sqrt{\frac{2d}{f}}$ = periodic time in the circle whose radius is d. And (Prop. XVII.) $\overline{2d}^{\frac{3}{2}} : 3.1416\sqrt{\frac{2d}{f}} : : z^{\frac{3}{2}}$: period. time in the ellipsis $= 3.1416\sqrt{\frac{2d}{f}} \times \overline{\frac{z}{2d}}^{\frac{3}{2}} = 3.1416 \times \frac{4ddf}{\overline{4df-vv}^{\frac{3}{2}}}$.

Cor. 2. *The* latus rectum *of the axis* AD *is* $= \frac{ppvv}{ddf}$.

Cor. 3. *Hence the transverse axis and the periodic time will remain the same, whatever be the angle of direction* SPT.

For no quantities but d, f, and v are concerned; all which are given.

Fig. 22.

SCHOLIUM.

Some people have dreamed that there may be a ſyſtem of a ſun and planets revolving about it, within any ſmall particle of matter; or a world in miniature. But this cannot be; for though matter is infinitely diviſible; yet the law of attraction of the ſmall particles of matter, not being as the ſquares of the diſtances reciprocally, but nearer the cubes; therefore the revolution of one particle of matter about another, cannot be performed in an ellipſis, but in ſome other curve; where it will continually approach to or recede from the center; and ſo at laſt will loſe its motion. Such motions as theſe can be nothing like that of a ſun and planets.

PROP. XXV.

24. *If a body revolves in the circumference of a circle* ZPA, *in a reſiſting medium, whoſe denſity is given. To find the force at any place* P, *tending to the center* C; *as alſo the time, velocity, and reſiſtance. Suppoſing the reſiſtance as the ſquare of the velocity.*

Draw PC, and *dp* parallel and infinitely near it, cutting the tangent P*d* in *d*. And put CZ = *r*, ZP = *z*, time of deſcribing ZP = *t*, velocity at P = *v*, reſiſtance = R, *f* = force at P, *g* = force of gravity at Z, *c* = velocity in Z. And let a body moving uniformly with the velocity 1, through the ſpace 1, in the time 1, meet the reſiſtance 1 in the medium. And let a body deſcend through the ſpace *a*, by the force *g* at Z, in the ſame time 1.

1. By the laws of uniform motion, the ſpace is as the time × velocity. Whence 1 (ſpace) : 1 × 1 (time × vel.) : : $\dot{z}$: $v\dot{t} = \dot{z}$, whence $\dot{t} = \frac{\dot{z}}{v}$.

2. By the nature of the circle, $dp = \frac{\dot{z}^2}{2r} = \frac{vv\ddot{t}t}{2r}$.

3. By

3. By accelerated motion, the ſpace is as the force × ſquare of the time; whence $g \times 1^2$ (force × time^2) : a (ſpace) : : $\ddot{ft}t$: dp or $\frac{vvtt}{2r}$: : $2rf$: vv = $\frac{2arf}{g}$. And $cc = 2ar$. Fig. 24.

4. The velocity generated (or deſtroyed) is as the force × time; therefore, $g \times 1$ (force × time) : $2a$ (velocity) : : $R\dot{t}$: $-\dot{v} = \frac{2aR\dot{t}}{g} = \frac{2aR\dot{z}}{gv}$, and $-v\dot{v} = \frac{2aR\dot{z}}{g}$.

5. The reſiſtance is as the ſquare of the velocity, whence 1^2 (vel.^{2}) : 1 (reſiſtance) : : vv : R = vv.

Therefore $-v\dot{v} = \frac{2aR\dot{z}}{g} = \frac{2avv\dot{z}}{g}$. And $-\frac{\dot{v}}{v} = \frac{2a\dot{z}}{g}$, whence $-\log. v = \frac{2az}{g}$, and corrected, $\log. \frac{c}{v} = \frac{2az}{g}$.

Again, ſince $\frac{2arf}{g} = vv$, $\frac{ar\dot{f}}{g} = v\dot{v} = -\frac{2avv\dot{z}}{g}$, and $\dot{f} = -\frac{2vv\dot{z}}{r} = -\frac{4arf\dot{z}}{gr}$, and $-\frac{\dot{f}}{f} = \frac{4a\dot{z}}{g}$, and $-\log. f = \frac{4az}{g}$, and corrected, $\log. \frac{g}{f} = \frac{4az}{g}$.

Alſo $\dot{t} = \frac{\dot{z}}{v} = \frac{-g\dot{v}}{2avv}$, and $t = \frac{g}{2av}$, and corrected $t = \frac{g}{2av} - \frac{g}{2ac}$.

Cor.

Fig. 24. Cor. 1. *Hence* $v =$ *number belonging to the logarithm log.* $c - \frac{2az}{g}$. *And* $f =$ *number belonging to the logarithm log.* $g - \frac{4az}{g}$.

Cor. 2. *Therefore the logarithms of v and f, each of them ſeverally decreaſes equally, in deſcribing equal ſpaces,* ad infinitum. *And therefore at every revolution, the log. of v is equally diminiſhed, and likewiſe that of f. But the body will revolve for ever, for when v is* 0, *t will be infinite.*

Cor. 3. *Hence if the central body at* C, *was ſo diminiſhed that its log. may decreaſe equally in deſcribing equal ſpaces, or in each revolution, after the manner as before mentioned; then the body will perpetually revolve in a circle, in a medium of uniform denſity.*

SECT.

Fig.

SECT. III.

The Motion of three Bodies acting upon one another; the perturbating Forces of a third Body. The Motion of Bodies round an Axis at reſt, or having a progreſſive Motion, and other Things of the ſame Nature.

PROP. XXVI.

IF a body be projected from A, *in a given direction* AD, *and be attracted to two fixed centers* S, T, *not in the ſame plane with* AD; *the revolving triangle* SAT, *drawn through the moving body, ſhall deſcribe equal ſolids in equal times, about the line* ST. 25.

Divide the time into infinitely ſmall equal parts; it is plain that equal right lines AB, BC, CD, &c. would be deſcribed in theſe equal times; and conſequently that all the ſolid pyramids STAB, STBC, STCD, &c. are equal, which would be deſcribed in the ſame equal times; if the moving body was not acted on by the forces S and T.

But let the forces at S and T, act at the end of the ſeveral intervals of time; as ſuppoſe the force T to act at B in direction BT; ſo that the body, inſtead of being at C, is drawn from the line BC, in the direction CF, parallel to BT. And in like manner it is drawn from the line BC, by the force S, in direction CE parallel to SB. And therefore, by the joint forces, the body at the end of the time, muſt be ſomewhere in the plane ECF parallel to SBT,

Fig. 25 SBT, as at I. But (Geom. VI. 17.) the ſolid pyramids STBI and STBC, are equal; being contained between the parallel planes ECF and SBT, and therefore have equal heights; whence STBI = pyramid STAB.

In like manner continue BI, making IK = BI; and in the next part of time, the body would arrive at K, deſcribing the pyramid STIR equal to STBI, But being drawn from the line IK, by the forces S, T, in the directions KL, KN, parallel to IS, IT; the body will be found at the end of the time, ſomewhere in the plane LKN parallel to SIT, as ſuppoſe at O, and then it will have deſcribed the ſolid STIO = STIK = STBI = pyramid STAB.

And in the ſame manner producing IO to P, till OP = OI. Then the body, attracted from O, by the forces S, T, will deſcribe another equal pyramid. And ſo it will continue to deſcribe equal pyramids in equal times; and conſequently the whole ſolids deſcribed are proportional to the times of deſcription.

Cor. 1. *When the number of* lineolæ. AB, BI, IO, *&c. is increaſed, and their magnitude diminiſhed,* ad infinitum; *the orbit* ABIO, *becomes a curve.*

Cor. 2. *Any line* AB *is a tangent at* A, BI *at* B, *&c.* A, B, *&c. being any points in the orbit.*

Cor. 3. *But the orbit* ABIO *is not contained in one plane, except in ſome particular caſes.*

For that the orbit may not deviate from a plane; the forces on both ſides thereof, ought to be alike.

Fig.

PROP. XXVII.

If the body T *revolves in the orbit* TH, *about the body* S *at a great diſtance, whilſt a leſſer body* P *revolves about* T *very near; and if* C *be the centripetal force of* S *acting upon* T. *Then the diſturbing force of* S *upon* P *is* $= \frac{3PK}{ST}C$. *Suppoſing* PK *parallel, and* KT *perp. to* ST. *And* $\frac{PT}{ST}C$ = *the increaſe of centripetal force from* P *towards* T. 26.

Let ST $= r$, PT $= a$, PK $= y$, g = force of gravity, h = ſpace deſcended thereby in time 1, s = the ſpace deſcended in the time 1, by the force C, p = periodic time of T about S, and t = per. time of P about T, γ = centripetal force of T at P, $\pi = 3.1416$.

Since attraction is reciprocally as the ſquare of the diſtance, then force of S acting at T : force of S acting at P : : $\frac{1}{ST^2} : \frac{1}{SP^2} :: \frac{1}{rr} : \frac{1}{\overline{r-y}^2} :: r :$ $r + 2y$, nearly. And force of S acting at T : to difference of the forces : : $r : 2y$; that is, $r : 2y ::$ $C : \frac{2y}{r}C$ = difference of the forces; and this is the ſingle force by which P is drawn from the orbit QAZ in direction KP or PS.

But ſince the motion will be the ſame, whether the ſingle force PS act in the direction PS; or the two forces PT, TS act in the directions PT, TS; ſubſtitute theſe two for that ſingle one; therefore proceeding as before, the force of S acting at T : force of S acting at P : : $\frac{1}{rr} : \frac{1}{\overline{r-y}^2}$. And force

of

Fig. 26. of S acting at P in direction PS : force of S acting on P in direction of TS : : PS : TS : : $r-y$: r : : $\frac{1}{r}$: $\frac{1}{r-y}$. Therefore *ex æquo*, force of S acting at T : force of S acting on P in direction TS : : $\frac{1}{r^3}$: $\frac{1}{r-y^3}$: : $\frac{1}{r^3}$: $\frac{1}{r^3-3r^2y}$: : $\frac{1}{r}$: $\frac{1}{r-3y}$: : r : $r+3y$, nearly. And the force at T : difference of the forces : : r : $3y$; or r : $3y$: : C : $\frac{3y}{r}$C = disturbing force of P, acting in direction parallel to TS. And PK *(y)* : PT *(a)* : : increase of the disturbing force in direction PK $\left(\frac{y}{r}C\right)$: $\frac{a}{r}$C, the addition of the centripetal force in direction PT. For when the disturbing force was $\frac{2y}{r}$C, there was no addition of centripetal force at T, but a diminution thereof; as appears by the following Corol.

Cor. 1. *The simple disturbing force, whereby* P *is drawn towards* S, *is* = $\frac{2y}{r}$C. *And the diminution of centripetal force of* P *towards* T, *is* = $\frac{v}{r}$C. *And the accelerating force at* P *in the arch* PA, *is* = $\frac{z}{r}$C. *Putting* z = *sine of* 2PQ, v = *versed sine of* 2PQ.

For let x = PK, and draw KI perp. to PT; then by similar triangles, PT *(a)* : PK *(y)* : : PK : PI : : force PK $\left(\frac{2y}{r}C\right)$: force in direction IP or TP = $\frac{2yy}{ar}C = \frac{v}{r}C$.

Also PT *(a)* : TK *(x)* : : PK : KI : : force PK (2y

$\left(\frac{2y}{r}C\right)$: force in direction KI or $PA = \frac{2xy}{ar}C =$ Fig. 26.

$\frac{z}{r}C$. By Trigon. B. I. Prop. II. Schol.

Cor. 2. *The diſturbing force at* P *is* $= \frac{q}{59\frac{1}{2}}\gamma$, *q being the ſine of the diſtance from the quadrature,* P *the moon,* S *the ſun.*

For (Prop. V.) $C = \frac{ttr}{ppa}\gamma$, and $\frac{3y}{r}C = \frac{3tty}{ppa}\gamma = \frac{3q\gamma}{178\frac{3}{4}}$ (becauſe $\frac{y}{a} = \frac{q}{1}$) $= \frac{q}{59\frac{1}{2}}\gamma$ nearly.

Cor. 3. *If* S *be the ſun,* P *a body in the equinoctial of the earth; the diſturbing force at* P *is* $= \frac{qg}{12852000}$.

For when P is at the moon's orbit, the force is $\frac{q}{59\frac{1}{2}}\gamma$; but $g = 60 \times 60\gamma$, or $\gamma = \frac{1}{3600}g$, therefore the force becomes $\frac{qg}{59\frac{1}{2} \times 3600}$, and at the earth is $\frac{qg}{59\frac{1}{2} \times 60^3}$.

Cor. 4. *If* S *be the moon,* P *a body on the equinoctial of the earth. The diſturbing force at* P *is* $= \frac{qg}{2880000}$.

For the general perturbating force was $\frac{3y}{r}C$, and here C muſt be the centripetal force at the moon. Now the centripetal force of the earth, at the diſtance of the moon is $\frac{1}{60^2}g$. And the moon being 40 times leſs than the earth, the centripetal force of

Fig. 26. of the moon, at the ſame diſtance, is $\frac{1}{40 \times 60^2} g$; put this for C, then the force of the moon upon the equinoctial, is $\frac{3y}{r} \times \frac{g}{40 \times 60^2} = \frac{3yg}{60a \times 40 \times 60^2}$

$= \frac{yg}{20 \times 40 \times 60^2}$

Cor. 5. *The diſturbing force of the ſun, to that of the moon, upon the equinoctial; is as* 1 *to* 4.46.

For theſe forces are as $\frac{1}{12852000}$ and $\frac{1}{2880000}$ or as 288 to 1285, or as 1 to 4.46.

Cor. 6. *If f be the apparent diameter, and d the denſity of the perturbating body. Then the diſturbing force will always be as* df^3y.

For that force is $\frac{3y}{r}$C or as $\frac{y\text{C}}{r}$. Let its diameter = b, M = its quantity of matter. Then C is as $\frac{\text{M}}{rr}$; that is, as $\frac{db^3}{rr}$. Therefore the diſturbing force is as $\frac{db^3y}{r^3}$, or as $dy \times f^3$.

Cor. 7. *If* P *be a point in the equator of the earth,* S *the ſun.*
The centrifugal force of P :
is to the perturbating force PT : :
As the ſquare of the earth's periodical time about the ſun pp :
to the ſquare of the earth's periodical time about its axis tt.

Let t = time of revolution of the earth round its axis; then t : $2\pi a$ (circumference) : : 1″ : $\frac{2\pi a}{t}$ = arch deſcribed in one ſecond; and the verſ-

ed

ed sine $= \frac{4\pi\pi aa}{tt \times 2a} = \frac{2\pi\pi a}{tt} =$ ascent or descent by the earth's centrifugal force. But forces are as their effects, whence $h : g :: \frac{2\pi\pi a}{tt}$ (ascent) $: \frac{2\pi\pi ag}{tth}$ the centrifugal force itself. But the perturbating force is $\frac{a}{r}C = \frac{asg}{rh}$. Whence the centrif. force : perturbating force $:: \frac{2\pi\pi ag}{tth} : \frac{asg}{rh} :: \frac{2\pi\pi}{tt} : \frac{s}{r} ::$ $2\pi\pi r : tts :: \frac{2\pi\pi r}{s} : tt$. But $\frac{2\pi\pi r}{s} = pp$. For $\sqrt{2rs}$: $1'' :: 2\pi r : p = \frac{2\pi r}{\sqrt{2rs}}$, and $pp = \frac{4\pi\pi rr}{2rs} = \frac{2\pi\pi r}{s}$ Fig. 26.

Cor. 8. *Hence the body* P *is accelerated from the quadratures* Q, Z, *to the siziges* A, B; *and retarded from the siziges to the quadratures. And moves faster, and describes a greater area, in the siziges than in the quadratures.*

PROP. XXVIII.

The same things supposed as in the last Prop. the linear error generated in P *in any time, is as the disturbing force and square of the time. And the angular error, seen from* T, *will be as the force and square of the time directly, and the distance* TP *reciprocally.*

For the motion generated in a given part of time, by any force, will be as that force; and in any other time as the force and the square of the time. The motion so generated is the linear error of P, as it is carried out of its proper orbit, by the force $\frac{3y}{r}C$. And that error, as seen from T, is as the

Fig. 26. angle it is seen under; and therefore is as that linear error, divided by the distance TP; and therefore is as the force and square of the time, divided by the distance.

Cor. 1. *The linear error generated in one revolution of* P, *is as the disturbing force and square of the periodical time* $\frac{3ca}{r}tt$. *And the angular error in one revolution is as the force and square of the periodic time divided by the distance.*

Cor. 2. *The mean linear error of* P *in any given time, will be as the force and periodical time,* $\frac{a}{r}$Ct. *And the mean angular one, as the force and periodical time, divided by the distance.*

For let the given time be 1; then t (time): $\frac{aC}{r}tt$ (whole error) : : 1 : $\frac{aCt}{r}$, the error in the given time.

Cor. 3. *The mean lineal error in any given time, is as* TP *and the periodical time of* P *directly, and the square of the periodical time of* T *reciprocally. And the mean angular error, as the periodical time of* P *directly, and the square of the periodical time of* T *reciprocally.*

For (Prop. V.) C is as $\frac{r}{pp}$, and $\frac{at}{r}$C is as $\frac{at}{r} \times \frac{r}{pp}$ or $\frac{at}{pp}$. And the angular error as $\frac{t}{pp}$.

Cor. 4. *In any given time, the lineal error is as* TP *and the periodical time of* P *directly, and the cube of* ST *reciprocally. And the angular error as the periodical time of* P *directly and the cube of* ST *reciprocally.*

For

For (Prop. XVII.) pp is as r^3, therefore $\frac{at}{pp}$ is as $\frac{at}{r^3}$. Fig. 26.

Cor. 5. *The linear error in a given time is as* $\frac{a^{\frac{5}{2}}}{r}$C, *and the angular error as* $\frac{a^{\frac{3}{2}}}{r}$C.

For t is as a^3, and $\frac{at}{r}$C as $\frac{a^{\frac{5}{2}}}{r}$C.

Cor. 6. *And universally, the angular errors in the whole revolution of any ſatellites; are as the ſquares of the periodic times of the ſatellites directly, and the ſquares of the periodic times of their primary planets reciprocally. And the mean angular errors are as the periodical times of the ſatellites, divided by the ſquares of the periodic times of their primary planets.*

For by Cor. 1. the angular error is as the force and ſquare of the time divided by the diſtance; that is, as $\frac{Ca}{r} \times \frac{tt}{a}$; that is, (becauſe C is as $\frac{r}{pp}$) as $\frac{tt}{pp}$. The reſt is proved in Cor 3.

PROP. XXIX.

If a ſpheriod AB *revolves about an axis* ST *in free ſpace, which axis is in an oblique ſituation to the ſpheriod; the ſpheriod will, by the centrifugal force, be moved by degrees into a right poſition ab; and afterwards by its libration, into the oblique poſition* αβ *And then will return back into the poſitions ab,* AB; *and ſo vibrate for ever.* 27.

Let C be the center of the ſpheroid; D the center of gravity of the end ICLB; E that of the end ICLA; D*d*, E*e* perp. to ST. Then the centrifugal force of the end CB, ſuppoſing it to act

Fig. 27. wholly at D, in direction *d*D, having nothing to oppoſe it, will move the end CB from B towards *b*, with a force which is as C*d*. And at the ſame time, the centrifugal force of the end CA, acting in direction *e*E, will move the end CA from A towards *a*, conſpiring with the motion of the end CB; by which means it will by degrees come into the poſition *ab*. And then by the motion acquired, it will come into the poſition $\alpha\beta$, making the angle $SC\alpha = SCB$. And the motion being then deſtroyed, it will return back, by the like centrifugal forces, acting the contrary way; and be brought again into the poſitions *ab*, and AB; and continue to vibrate thus perpetually.

Cor. *Hence if the axis of the earth is not preciſely the ſame as the axis of its diurnal rotation; the earth will have ſuch a libration as is here deſcribed, but exceeding ſmall. This is ſuppoſing it a ſolid body; but if it was a fluid, it would by the centrifugal force, form itſelf into an oblate ſpheroid.*

PROP. XXX. *Prob.*

28. *If a globe* APBQ *in free ſpace, revolve about the axis* SCT, *in direction* ADB; *and if any force applied at* V, *the end of the radius* CV, *acts by a ſingle impulſe in direction* VG *perp. to* CV, *in the plane* VCD. *To find the axis about which the globe ſhall afterwards revolve.*

Suppoſe the great circle VBQA perp. to the line of direction VG; and if VH, VI be 90 degrees; it is plain, if the firſt motion was to ceaſe; the globe by the impulſe at V, would revolve round the axis IH, which by the firſt motion was round the axis PQ. Therefore by both motions together it will move round neither of them. Now ſince

ſince a point of the ſurface moving with the greateſt velocity about ST, will move along the great circle ADB; and a point having the greateſt velocity about IH, moves along the great circle VDE. Therefore a point that will have the greateſt velocity, by the compound motion, will alſo be in a great circle paſſing through D. Therefore in the great circles DB, DE take two very ſmall lines D*r*, D*o*, in the ſame ratio as the velocities in AD, and VD; and compleat the parallelogram D*opr*. Then through D and *p* draw a great circle KD*p*L; and a point having the greateſt motion, ariſing from a compoſition of the other two motions, will move along KD*p*L. Therefore finding F, R, the poles of the circle KD*p*L, FR will be the new axis of revolution, or the axis ſought. And the velocity about the axis FR will be proportional to D*p*; VBQA being always ſuppoſed perp. to GV, or to the plane DVC. Fig. 28.

Note, if you ſuppoſe an equal force applied at E, in direction contrary to GV, it will by that means keep the center C of the globe unmoved, and will likewiſe generate twice the motion in the globe.

Cor. 1. *The greater the force is that is applied to* V, *the greater the diſtance* PF *is, to which the pole is removed. And if ſeveral impulſes be made ſucceſſively at* V, *when* V *is in the circle* APB, *the pole* F *will be moved further and further towards* H, *in the circle* APB.

For ſeveral ſmall forces or impulſes have the ſame effect as a ſingle one equal to them all.

Cor. 2. *If the force act at* P, *in direction perp. to the plane* CPB; *and* Dr, Do *be as the velocities along* DB *and* DQ. *The great circle* KDL (*paſſing through* D *and p) is the path of the point* D; *and its pole* F, *or axis of revolution* RF; *the pole being tranſlated* 29.

Fig. 29. *from* P *to* F. *And if the impulſe be exceeding ſmall,* PF *will alſo be exceeding ſmall.*

Cor. 3. *If the force at* P *always acts in parallel directions, whilſt the globe turns round. The pole will make a revolution in a ſmall circle upon the ſurface of the globe, in the time of the globe's rotation, and the contrary way to the globe's motion.*

For let a ſingle impulſe at P tranſlate the pole to F; and afterwards when the globe has made half a revolution, and the point P is come to *p*; then if a new impulſe be made at F, the pole will be tranſlated to *p* which is now P; that is, it will be moved back to its firſt place on the globe. So that in any two oppoſite points of rotation, the place of the pole is moved contrary ways, and ſo is carried back again the ſame diſtance. And ſince the globe revolves uniformly, if the force act uniformly, it will move the pole all manner of ways, or in all manner of directions upon its ſurface; that is, it will deſcribe a circle, which will end where it begun. And in deſcribing this polar circle, the motion will be contrary to the motion of the globe; for ſuppoſe PFB an immovable plane. If the globe ſtood ſtill, the pole would move in a great circle, in the plane PFB. But ſince all the points of the globe which come ſucceſſively to the plane PB, are not yet arrived at it, but are ſo much further ſhort of it, as PF is greater; it is evident all theſe points will lie on this ſide the plane PB. And as any fixed point will deſcribe a circle on the moving globe, contrary to the motion of the globe; ſo will a point that is not fixed, but moving in the plane PFB, likewiſe deſcribe a circle (or ſome curve) contrary to the motion of the globe. Or ſhorter thus, ſuppoſe the globe to ſtand ſtill, and the direction of the force to move backward, then the relative motion will be the ſame as before; and then

then the pole F will move backward too, as it will follow the force, being at right angles to it. Fig. 29.

Cor. 4. *Since the pole by one impulſe is tranſlated to* F; *the new pole* F *is therefore another point of the material globe, diſtinct from* P. *And the particle* P *that was before at reſt, will now revolve about the particle* F *at reſt.*

For the new pole F is that particle of the globe which happened to be revolving in F, when the impulſe was made at P. The matter of the great circle ADB does not come into the circle KDL, but only the point D of it. For when the force is impreſſed, the other particles M, N, by the compound motion, will be made to revolve in the directions M*m*, N*n*, parallel to D*p*; and therefore will deſcribe leſſer circles about F; whilſt only D deſcribes the great circle KDL.

Cor. 5. *What is demoſtrated of a ſphere is true alſo of an oblate ſpheroid, whoſe axis is* PQ; *and the force impreſſed at* P, *acting in direction perp. to* CPB, *or parallel to* CD *the radius.*

Cor. 6. *But if the force at* P *act in direction contrary to the foregoing (as in caſe of an oblong ſpheroid); the pole of rotation will be moved from* P *towards* A, *contrary to the way of the other motion.*

Cor. 7. *And in general the pole* P *will always be moved in a direction perp. to that of the power; and towards the ſame way as the ſpheroid revolves.*

Cor. 8. *Hence after every half rotation of the globe round its axis, the places upon the globe change their latitude a little; which, after an entire rotation return to the ſame quantity. But this variation is ſo trifling, as to come under no obſervation.*

This is evident, becauſe the pole is altered; and

Fig. 29. of conſequence, the diſtance therefrom is altered in all places, except in the great circle FCR.

PROP. XXXI. *Prob.*

30. *Let* AB *be an oblate ſpheroid, whoſe axis is* PC; *and let it revolve round that axis, in the order* ADB, *which is its equinoctial; and if any force act at* P, *in direction* PG *perp. to* PC, *and in the plane* P⊙C, *which moves ſlowly about, in the order* ADB. *To find the motion generated in the ſpheroid.*

Let EL be an immovable plane like the ecliptic, in which the center C of the ſpheroid always remains. ON another plane parallel to it. Erect CM perp. to theſe planes; and make the angle MCN = the angle B♎C, in which the equinoctial B♎ cuts the ecliptic CL. Suppoſe the ſpherical ſurface OVN to be drawn, whoſe pole is M; and produce CP to cut it at R, in the circle ORN. Then PCD and RCM are in one plane, and both of them perp. to P♎C and P♈C. Now to find the motion of the axis of the ſpheroid. Here OVN is the upper ſide of the ſurface.

This Prop. differs from the laſt in ſeveral reſpects. The laſt Prop. regards only the motion of the pole upon the ſurface of the globe, and that is cauſed by a motion which is generated in the globe itſelf. But in this Prop. we conſider the motion of the axis CR in the fixed ſpherical ſurface OVN; which always proceeds in one direction, as long as the moving force keeps its poſition. In the laſt Prop. the motion of the globe round its axis is performed in a very ſhort time; but here the revolution of the force, in the moving plane P⊙C, is a long time in its period.

Now by the laſt Prop. Cor. 7. the force PG will always move the axis of rotation CPR in a direction

tion perp. to PG. Therefore ſuppoſe the plane P⊙C to revolve ſlowly round PC, and we ſhall find that in the beginning of the motion, when ⊙ or F is at ♎, then the plane (P⊙C or) P♎C will be perp. to the plane CRM, and at that time the motion of R will be directed from R towards M. And when that plane comes to the poſition PF⊙C, the motion of R will be directed to ſome place between N and V. And when it is got to the tropic D, then R's motion is directed along the circumference RV; for then P⊙C coincides with CRM. But when P⊙C arrives at *f*, the motion will be directed from R to ſome point *t* without the circle. And laſtly, when P⊙C is at the other interſection ♈, beyond B; the motion of R will be directed to *m* oppoſite to M. The reſult of all which is, that the pole R will deſcribe ſuch a curve as R1234; and then the ſame force begins again at ♎; which being repeated, another ſimilar figure 456 is deſcribed by R; and ſo on for more. The ſame force I ſay is repeated, for when the plane P⊙C comes on the other ſide of the globe, the force acts the contrary way, and therefore it is all one as if it acted on the firſt ſide of the globe. Fig. 30.

It muſt be obſerved, that as R moves through 1234, the interſection ♎ gradually moves towards E. And as to the force PG, it may be ſuppoſed variable, at different poſitions of the plane P⊙C. And according to the quantity of force in the ſeveral places, different curves (1234) will be deſcribed.

Cor. 1. *Hence it is evident, that the inclination of the planes* ADB *and* ECL, *is greateſt when* P⊙C *paſſes through* ♎ *and* ♈. *And leaſt when it paſſes through the tropic* D. *And that the inclination decreaſes from the node* ♎ *to the tropic* D, *and increaſes from the tropic to the node.*

For

Fig. 30 For when P⊙C is in ♎, R is fartheſt from M; but when it is in D, R is at 2, and its direction is parallel to R4, and then 2M is leaſt.

Cor. 2. *After a revolution of the plane* PF⊙C *(in which the force always acts), the inclination comes to the very ſame it was at firſt.*

For at any two points F, *f*, equidiſtant on each ſide from the tropic; the force is directed contrary ways, from and to the circle RV; and therefore the motion, in the curve R1234, being alſo equal and contrary, from and towards RV, they mutually deſtroy one another; and therefore after a revolution, or rather half a revolution, the pole R is brought back to the circle RV, and then the angle RCM is the ſame as it was at firſt.

Cor. 3. *The motion of the pole* R, *reckoned in the circle* OVN, *is always from* R *towards* V, *then through* N, O, *and* R.

For though the motion of R towards and from M, in the line M*m*, in one revolution, is equal both ways; and ſo R is always brought to the circle again; yet the motion conſidered along the circle is always in the order RVN. Thus it goes through the curve 123 to 4, ſo that after half a revolution of P⊙C, it is advanced forward in the circle RV, the length R4.

Cor. 4. *The motion along the circle is ſometimes faſter and ſometimes ſlower. At* 2 *it moves faſteſt of all; at* R *and* 4, *it moves ſloweſt, or rather is ſtationary for a moment.*

Cor. 5. *The pole* R, *and the nodes move the contrary way about, to what* AB *revolves.*

Cor. 6. *If the force* PG *ſtands ſtill, the pole* R *will ſtill move backwwards as before; and that in a right*

right line, or rather, a great circle. And if PG *moves backward through* BDFA; *the pole* R *will ſtill go backward; but then the curve* R24, *will be concave towards* M, *like* 87R, *being contrary to the other where the force moved forward.* Fig. 30.

Cor. 7. *But in an oblong ſpheroid, where the force acts in direction* GP, *quite contrary to the other;* R *will deſcribe the curve* R78, *without the circle* ORV; *every particle of it in a contrary direction to theſe of* R24. *And therefore the pole* R, *and the nodes* ♈ *and* ♎ *will move the ſame way about as* ADB *revolves, and contrary to what they do in an oblate ſpheroid.*

For the force being directed the contrary way; of conſequence the motion muſt be ſo too.

Cor. 8. *And in an oblong ſpheroid, if the force* GP *move the contrary way about; yet the pole* R *will ſtill move forward. And the curve deſcribed by* R, *will have its convexity the contrary way.*

Cor. 9. *Hence if the quantities and proportions of theſe forces, in different places be known; it will not be difficult to delineate the curve* R1234, *upon the ſpherical ſurface* OVNM.

PROP. XXXII. *Prob.*

If a planet (or the moon) move in the orbit ATE*t round an immovable center* C, *whoſe plane is inclined to the plane of the ecliptic* AQE; *and a force acts upon it in lines perp. to* GZ, *and parallel to the ecliptic, directed always from the plane* GZ *to either ſide. To find the motion of the nodes* A, E; *and the variation of the orbit's inclination* PAO. 31.

Let ATZE be half the orbit raiſed above the ecliptic AQ, AE the line of the nodes; T, *t*, the

Fig. 31. the tropics. Draw CM perp. to the ecliptic AQE, and CR perp. to the orbit ATE. Round M as a pole deſcribe a ſpherical ſurface RVNX; then RC will be the axis of the orbit, and MC of the ecliptic, or circle AQE. Through the points T, C, M, draw the plane RMC; and through A, C, M, another plane cutting the circle RNF in X and H; then RF is perp. XH, and the circle RNX parallel to GEQ. Note, RNX is the upper ſide of the ſurface.

Let the planet be at P, and let P2 be the ſpace it deſcribes in any ſmall time, and the line P1 the ſpace it would be drawn through in the ſame time, by the force acting from the plane MGZ. Compleat the parallelogram P123, and P3 will be its direction by the compound force. Now as the line P1 is parallel to the ecliptic, it is plain the point 3 will be below the plane of the orbit; and the plane CP2 will be moved into the poſition CP3, revolving about CP; conſequently the axis RC will be moved in a direction perp. to CP. And the pole R will be moved to ſome point between F and H. This being duly attended to, the motion of the pole R will be known for all the places of P in the orbit GATZ. For about G the motion of R is directed perp. to M*n*; at A it moves perp. to MX, or in direction RM. At T it moves parallel to MH, or in the curve RV. Approaching to Z, it moves perp. from M*n*. So that in the paſſage of the planet P, from G to Z, through GAPTZ, the pole R of its orbit, moves through the curve R1234. But in the other half of the orbit ZE*t*G, as the force is directed the contrary way from the plane GZMN, the pole R will return back at 4, and deſcribe a ſimilar curve 45678. So that when the planet P has made one revolution, the pole of its orbit R will be found at 8. But in this poſition of the nodes, the point 8 will be within the ſpherical

rical ſurface RHNX, not reaching the periphery RV. For the ſeveral parts of the curve being deſcribed in all directions in reſpect of the line *n*N; the points R, 4, will be equidiſtant from *n*N, and likewiſe 4, 8, for the ſame reaſon. Fig. 31.

Now ſuppoſe the force and the plane GZN*n* to revolve about the axis CM in the order GATE. Then after it is come to ſuch a poſition, that the aſcending node A is as far on the other ſide of G, ſuppoſe at *a*; then the pole R will be as far on the other ſide of V, ſuppoſe at *r*; and being alſo as far from N*n*, on the ſame ſide, the curves (12468) will approach VH there, by the ſame degrees as they receded from it at RV. And therefore the pole R will by degrees be brought to the circle again. Thus in every two correſpondent points on each ſide V, the forces and their effects balance one another, and R will be at the ſame diſtance from the circle RVH. And therefore after half a revolution of the plane GZ to the nodes, the angle RCM, and conſequently the inclination of the orbit, comes to the ſame as at firſt. And likewiſe as the pole R moves forward or backward in the circle RVH, the motion of the nodes A, E, will be forward or backward.

Cor. 1. *In this poſition of the nodes at* A *and* E, *the inclination of the orbit* ATE *will be diminiſhed every revolution of* P. *But on the oppoſite ſide at a, the inclination increaſes every revolution of* P.

For the points 4, 8, come nearer and nearer to M, and the contrary at *r*.

Cor. 2. *When the nodes are at* A, E; *the inclination decreaſes*; *when the planet is in* GT *or* Z*t*; *and increaſes in* TZ *and t*G.

For R moves to 3, whilſt P moves through GT. At 3 it is at its neareſt diſtance to M; from 3 to

Fig. 31. 4, R recedes from M, whilſt P moves through TZ. And the like on the other half of the orbit.

Cor. 3. *When the planet is in* GA *and* ZE, *the nodes go forward. But in* AZ *and* EG, *they go backward.*

For whilſt R paſſes through R1, its motion is forward, viz. from R toward *n*, and at 1 where it moves parallel to RM, it is ſtationary; that is, when P is in A. Through 1234, R moves backward, or towards V; and then P is in AZ.

Cor. 4. *In general, the nodes are always regreſſive, except when* P *is between a node, and its neareſt quadrature; and then they are progreſſive, wherever the nodes are ſituated.*

Cor. 5. *The nodes go faſteſt back when the planet is in* T *and t.*

For then R is at 3 and 7.

Cor. 6. *The inclination varies moſt, when* P *is at* A *and* E.

For then R is at 1 and 5.

Cor. 7. *And from the various ſituations of the nodes, and the place of* P, *it may eaſily be determined, when the inclination increaſes or decreaſes, in any caſe.*

Cor. 8. *Hence if the quantities of theſe forces were known, it would not be difficult to delineate the motion of the pole* R, *upon the ſpherical ſurface* RXFH; *and at any time to find the inclination, and place of the node.*

Cor. 9. *And to find the nature of the curve* R1234 *deſcribed by the pole* R. *Snppoſing the force directed always to the ſun; and to be as the diſtance of* P *from the plane* GZ.

32. Ler RDB be the curve, and let the tangent *t*DT revolve about the curve RDB, begining at R, ſo

as

as the end T may move uniformly through all the points of the compaſs, in the ſame manner as P moves through its orbit ATZE*t*. It is plain this is one property of the curve RDT. Fig. 32.

Now ſince the ſun's rays fall at the ſame obliquity upon all parts of the plane ATQ, therefore the force to draw P in a direction parallel to theſe rays, being the ſame at equal diſtances from the plane GZ, and always as the diſtance; therefore by the reſolution of motion, the diſtance that P is drawn perpendicularly from the plane of its orbit, will alſo be as that diſtance; and that is as the variation of the orbit's inclination. Therefore if P, inſtead of moving to 2 move to 3, then the force at P or PB (fig. 31.) will be as the angle 2P3: ſuppoſing the ſun's diſtance from the node to remain the ſame, during one revolution of P. 33.

But when the ſun or the force alters its poſition, it will be greater or leſs on that account, in proportion to the ſine of OL (where OL is perp. to AL), and that is as the ſine of AO, the diſtance from the node, the angle A being given. From hence it follows that univerſally, the force acting on P will be always as BP × S.AO; that is, as S.GP × S.AO (fig. 31.); that is, as the ſine of the diſtance of P from the quadratures, and the ſine of the diſtance of the ſun at O from the node. 34.

Now let us find the nature of the curve R1234, ſuppoſing AO to remain the ſane for one revolution of P. Put RA = x, AD = y, RD = z. Since by the generation of the curve, the angle AD*t* = arch GP, and the force is as the ſine of GP or of AD*t*, and $\frac{\dot{x}}{\dot{z}}$ = S.AD*t*. Alſo it is plain, the increment of the curve at D is as that force; therefore $\frac{\dot{x}}{\dot{z}}$ is as $\dot{z}$. And ſince in paſſing through the 32.

particle

Fig. 32. particle of the curve z, the line Dt is ſuppoſed to change its direction uniformly, therefore the angle of contact is given; whence $\dot{z}$ or $\frac{\dot{x}}{\dot{z}}$ is as the radius of curvature, or as $\frac{\dot{z}\dot{y}}{\ddot{x}}$; that is, $\frac{\dot{x}}{\dot{z}}$ is as $\frac{\dot{z}\dot{y}}{\ddot{x}}$, or $\dot{a}\dot{x}\ddot{x} = \dot{z}^2\dot{y}$ ($\dot{z}$ being given), and the fluent is $\frac{\dot{a}\dot{x}^2}{2} = y\dot{z}^2$, and $\dot{x} : \dot{z} :: \sqrt{y} : \sqrt{\frac{1}{2}a} ::$ as the ſubtangent : to the tangent; which is the property of the cycloid, $\frac{1}{2}a$ being = CB, the diameter of the generating circle.

Now at different diſtances of O from the node, the cycloid deſcribed wtll be greater in proportion to the ſine of AO (fig. 31.); and even in the ſame cycloid, the latter part will be greater than the former part, as AO grows greater; all the parts of it increaſing as the ſine of AO increaſes; and the greateſt cycloid will be when A is in the quadratures; and the leaſt when in the ſyziges, where it is reduced to nothing.

Scholium.

From the foregoing ſolution, theſe obſervations may be made.

1. Though the curve R24 has been determined to be a cycloid, yet it is nearer an epicycloid. For at R it ſets off nearly in a direction perp. to GZ, and during its generation (that is, whilſt P performs a ſemirevolution) the point A moves towards G; and ſuppoſing the force at O to be fixed, the laſt particle of the curve at 4 would be parallel to that at R. But as O really moves forward, ſome number of degrees, ſuppoſe 14, and continues to do ſo, all the ſemirevolution; therefore every particle of the curve will have other directions

rections in its deſcription, being more curve than before; and at laſt the tangents at R and 4, will make an angle of 14 deg. which is the ſame as if an epicycloid was deſcribed on the convex ſide of a circle, going through an arch of it equal to 14 degrees. Fig. 34.

2. Thus the curve deſcribed by R would be nearly an epicycloid, when the force at O is every where of the ſame quantity; yet as O moves about, the force will increaſe and decreaſe in proportion to the ſine of AO; therefore, if you will ſuppoſe ſuch an epicycloid deſcribed as above mentioned, and moreover imagine the radius of the generating circle to ſwell or increaſe, in the ſame ratio as the S.AO increaſes; then ſuch an epicycloid will nearly repreſent the curve deſcribed by R. For then every part of it will be greater or leſs, in proportion to the force that generates it. But enough of this. All that I ſhall add on this head is the ſolution of the two following problems, upon account of their curioſity, as depending on the foregoing principles.

PROP. XXXIII. *Prob.*

To find the diſturbing force of Jupiter or Saturn, upon the earth in its orbit; having that of the ſun upon the moon given.

Let the matter in the ſun and Jupiter be as *m* to 1. E, I, L, the periodic times of the earth, Jupiter and the moon. A, B, the diſtances of the earth and Jupiter from the ſun. D the moon's diſtance from the earth. C, *c*, the centripetal forces of the ſun and Jupiter. 26.

Then (by Prop. XXVII.) the diſturbing force of S upon P, is $\frac{3PK}{ST}C$ or as $\frac{PT}{ST}C$. Therefore if S be the ſun, and P the moon, the diſturbing force

Fig. 26. is $\frac{D}{A}C$; but if S be Jupiter, P the earth, T the ſun; then the force is $\frac{A}{B}c$. That is, the ſun's diſturbing force upon the moon is to Jupiter's diſturbing force upon the earth; as $\frac{D}{A}C$ to $\frac{A}{B}c$; or as DBC to A^2c. But (Cor. 2. VII.) $C=\frac{m}{AA}$, and $c=\frac{I}{BB}$. Therefore the ſun's force upon the moon, is to Jupiter's upon the earth; as $\frac{DBm}{AA}$ to $\frac{AA}{BB}$, or as DB^3m to A^4; that is (Prop. XVII.), as DI^2m: AE^2. That is, the ſun's diſturbing force upon the moon, is to Jupiter's diſturbing force upon the earth; as $D\times II\times m$, to $A\times EE$. But that of the moon is known, and conſequently that of Jupiter. And if for I and m, we put Saturn's periodic time, and quantity of matter; Saturn's diſturbing force will be known.

Cor. 1. *The angular errors generated in the moon by the ſun, are to the errors generated in the earth by Jupiter in the ſame time, : : as* $IIL\times m$, *to* E^3.

For (Prop. XXVIII. Cor. 2.) theſe errors are as the forces and periodic times, divided by the diſtances. Therefore the ſun's effect to Jupiter's, is as $\frac{D\times II\times m\times L}{D}$ to $\frac{A\times E^2\times E}{A}$; or as $IILm$ to E^3.

Cor. 2. *Hence the error generated in the moon by the ſun, is to the error generated in the earth by Jupiter; as* 11230 *to* 1, *and to that generated by Saturn, as* 196076 *to* 1.

For put $I=4332\frac{1}{2}$ days, $L=27\frac{1}{3}$, $m=1067$, $E=365\frac{1}{4}$; then $\frac{I^2Lm}{E^3}=11230$. And putting

I =

$I = 10759\frac{1}{4}$, and $m = 3021$, for Saturn; then $\frac{I^2 L m}{E^3} = 196076$. Fig. 26.

Cor. 3. *The force of Saturn to the force of Jupiter to disturb the earth, is as* 1 *to* $17\frac{1}{2}$.

Cor. 4. *The motion of the nodes of the earth's orbit by Jupiter's action, in* 100 *years, is* $10' 20''\frac{1}{2}$. *And by Saturn's,* $35''\frac{2}{3}$.

For the motion of the moon's nodes in a year is $19° 20' 32''$, or $69632''$; this divided by 11230 gives $6''.2005$, multiplied by 100, is $620''.05$, which increased in the ratio of the cosine of inclination of Jupiter's orbit ($1° 19' 10''$), to that of the moon's ($5° 8'\frac{1}{2}$), produces $10' 22''\frac{1}{2}$. Which diminished in the ratio of 1 to $17\frac{1}{2}$, gives $35''\frac{2}{3}$ for Saturn.

Cor. 5. *The motion of the earth's aphelion by the action of Jupiter, is* $21' 44''$ *in* 100 *years, in consequentia. And by Saturn,* $1' 14''\frac{1}{2}$.

For the motion of the moon's apogee is $40° 40' 43''$, or $146443''$ in a year. This divided by 11230 gives $13.04''$; which multiplied by 100, gives $1304''$ or $21' 44''$. And divided by $17\frac{1}{2}$, gives $74''\frac{1}{2}$.

PROP. XXXIV. *Prob.*

To find the variation of inclination of the earth's orbit, by the action of Jupiter in 100 *years; and the like for Saturn.* 35.

Let ♈ ♋ ♎ ♑ be the ecliptic, or plane of the earth's orbit; GFH the orbit of Jupiter; G Jupiter's ascending node; E, I, Q, the poles of the ecliptic, Jupiter's orbit, and the equator, respectively; ECK a circle parallel to GF; and D*m*Q a circle parallel to the ecliptic. The pole Q here moves regularly along the circle Q*l*D, by the pre-

Fig. 35. ceſſion of the equinoxes; which circle is no way affected or altered by Jupiter's action; becauſe Jupiter cannot be ſuppoſed to have any force to move the equinoctial points, or alter their regular motion. But he has a power of acting upon the whole body of the earth, and altering its orbit, and conſequently the pole E of the ecliptic; which pole is therefore made to move along the circle ECK. Therefore we muſt ſuppoſe the orbit of Jupiter fixed, and conſequently the pole I, and circle ECK. And now we have to compute the motion of E along the circle ECK.

The preceſſion of the equinoxes in
100 years is — — 1° 23′ 20″
And (by the laſt prob.) the motion of
Jupiter's nodes in 100 years is 10′ 22″½ or 622″½
Jupiter's aſcending node G (1755,
angle QEG) -- — ☋ 8° 20′
Inclination of Jupiter's orbit — 1° 19′ 10″.
Therefore make the angle QE*a* = 1° 23′⅓, and EIC = 10′ 22″½. Upon E*a* let fall the perp. C*o*; then E*o* is the decreaſe of EQ or E*a*, which (Prop. XXXII.) is the ſame as the decreaſe of the inclination of the planes of the ecliptic and equinoctial.

In the triangle EIC, by reaſon of the very ſmall angle EIC, we ſhall have as rad : S.IC (1° 19′ 10″) : : angle EIC (622″½) : EC = $\frac{\text{EIC} \times \text{S.IC}}{\text{rad}}$ = 14″.3. To the angle GEQ (8° 20′), add QE*a* (1° 23′⅓), then *a*EG or *o*EC = 9° 43′⅓. Then in the very ſmall right lined triangle EC*o*, rad: EC : : cof. *o*EC (9° 43′⅓) : E*o* = $\frac{\text{EC} \times \text{cof.}\,o\text{EC}}{\text{rad}}$ = 14″.1.

Or E*o* = $\frac{\text{EIC} \times \text{S.IC} \times \text{cof.}\,o\text{EC}}{\text{rad}^2}$, the decreaſe of inclination of the equinoctial in 100 years by the action

action of Jupiter. And this decreaſe will amount to a minute in 425 years. Fig. 35.

If the ſame computation was applied to Saturn, putting EIC = $35'' \frac{2}{3}$, IC = 2° 30′ 10″, *o*EC = (21° 21′ 36″ + 1° 23′ $\frac{1}{3}$) 22° 44′ 56″; the decreaſe by Saturn will be 1″.44. Therefore the decreaſe by both will be 15″.54; which will be a minute in 386 years.

Cor. 1. *The inclination will decreaſe till* E *and a be at their neareſt diſtance in the two circles, which will be above* 6000 *years; and then it will increaſe again. It has likewiſe been decreaſing for above* 8000 *years.*

For the diameters of the circles EK and DQ, being about as 1 to 17. And the angle IEQ being 81° 40′, and the difference of the motions of E and Q being 1° 13′; it will decreaſe nearly as many centuries as is the quotient of 81° 40′ divided by 1° 13′, which is 67. Alſo the ſupplement 98° 20′ divided by 1° 13′, gives 81 centuries, it has been decreaſing.

Cor. 2. *But the increaſe or decreaſe for every century is not* 15″.54, *as determined in this particular ſituation. For as it approaches to its maximum or minimum, it varies very ſlow, and at theſe places is at a ſtand for a long time.*

Cor. 3. *The inclination can never be leſs than* 20° 50′ 54″; *nor greater than* 26° 7′ 20″.

For the neareſt and greateſt diſtances of the two circles EK, DQ amount but to theſe. And there muſt be many revolutions, before they can light upon theſe two points, if the world can be ſuppoſed to exiſt ſo long.

Scholium.

The diſturbing forces of Jupiter and Saturn here made uſe of are derived from that of the ſun up-

Fig. 35. on the moon; but these forces are really more than are here determined. For in calculating the disturbing force of the sun (by Prop. XXVII.), the forces upon T and *p* are as *rr* to $rr + 2ry + yy$; but by reason of the great distance of the sun, the part *yy* is left out, as being extremely small. But in the case of Jupiter and Saturn it is otherwise, and therefore *yy* must be taken in. Whence the disturbing force must have an additional increase, which is as $2ry$ to *yy*, or as $2r$ to *y*, which in Jupiter is as 10 to 1, and in Saturn as 19 to 1. Therefore Jupiter's disturbing force must be increased by $\frac{1}{10}$th, and Saturn's by $\frac{1}{19}$th; which being done, their effects will be proportional, and the decrease of inclination of the ecliptic in 100 years, by Jupiter and Saturn will become 15″.45, and 1″.51; and by both 16″.96 or 17″ nearly; which will amount to a minute in 353 years.

But if the observations of the antients can be depended on, the obliquity decreases faster than this. For by the observations of *Aristarchus*, *Eratosthenes*, *Hyparchus*, *Ptolemy*, and *Theon*, the obliquity was found to be 23° 51′. And none of them lived 300 years before Christ, and two of them after. So that in little more than 2000 years, there is a difference of 22′; which is more than a minute in 100 years, and is more than three times as fast as we have here determined it.

I shall now proceed to some things of another kind, relating to centripetal forces; which as far as I can find, have not been meddled with by any body before.

PROP. XXXV. *Prob.*

36. *If the circle* GDFE *be moved along the right line* AB, *whilst it turns round its axis; to find its motion upon a horizontal plane.*

Suppose the circle inlined in any given angle to the horizon, the line of direction AB being at first in

in its plane; and let it move round its axis CP, which is perpendicular to its plane, with any velocity. Let O be the center of oſcillation. Fig. 36.

Now the circle will endeavour to deſcend by its gravity, in the ſame manner, as the ſingle point O would do. Therefore ſuppoſe the point F to deſcend through F*f* in a moment of time, and that F is transferred to I in the ſame time. Then if the parallelogram F*f*nI be compleated, the point F by the compound motion, will move along F*n*. By this means CP, the axis of revolution, will be transferred to the poſition C*q*, inclining more towards B. But when the circle has made half a revolution, and G is come to the place F; the points in F, proceeding in the track F*n*, will move the axis C*q* forwards, as before; that is (by the turning of the circle) it will throw it into its former place CP. So that during a revolution, the axis is thrown contrary ways in all the oppoſite points, and ſo is always reſtored back to its firſt place in regard to the plane. Therefore the circle always revolves about the axis CP, whilſt CP continually inclines more and more forward; that is, the plane of the circle continually alters its poſition; and the variation of its poſition is known from the lines FI, *n*I; and is equal to the angle *n*FI or PC*q*. And ſince the circle endeavours to move along a line which is in the plane *n*FG, it will no longer go along AB, but deviate from it into a new track, which is now to be found.

It may be convenient to imagine the circle to be a poligon with an infinite number of ſides. Then let KL be the horizon, M*g* the plane of the circle, *g* being a point in the right line AB; let the next point of the circle (or angle of the poligon) deſcend to the horizon, through the very ſmall ſpace *rt*; then in the right angled triangle *trg*, S. inclination (*tgr*) : *tr* or *n*I (which is as S.*n*FI) : :

Fig. 35. on the moon; but these forces are really more than are here determined. For in calculating the disturbing force of the sun (by Prop. XXVII.), the forces upon T and p are as rr to $rr + 2ry + yy$; but by reason of the great distance of the sun, the part yy is left out, as being extremely small. But in the case of Jupiter and Saturn it is otherwise, and therefore yy must be taken in. Whence the disturbing force must have an additional increase, which is as $2ry$ to yy, or as $2r$ to y, which in Jupiter is as 10 to 1, and in Saturn as 19 to 1. Therefore Jupiter's disturbing force must be increased by $\frac{1}{10}$th, and Saturn's by $\frac{1}{19}$th; which being done, their effects will be proportional, and the decrease of inclination of the ecliptic in 100 years, by Jupiter and Saturn will become 15″.45, and 1″.51; and by both 16″.96 or 17″ nearly; which will amount to a minute in 353 years.

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Fig. 36. rad (1) : $tg = \frac{S.nFI}{S.tgr}$. Therefore if the circle be moved through Gg in the ſame time, then from g (in the line AB) ſetting off $gt = \frac{S.nFI}{S.\text{inclination}}$, then t will be a point in the curve GtR through which the circle will paſs. After the ſame manner it will again deviate from the laſt direction tb; and deſcribe the curve GRVWX.

Generally the whirling motion, round its axis, is equal to its progreſſive motion; for the friction of the plane ſoon reduces it to that. But take away the friction, and theſe two different motions may be what you will.

Cor. 1. *The curvature in any place, is reciprocally as theſe three quantities, the velocity of rotation, the progreſſive velocity, and the tangent of inclination.*

For the curvature is as the angle tGg, that is, as $\frac{tg}{Gg}$ or as $\frac{S.nFI}{Gg \times S.\text{inclination}}$; that is, as $\frac{nI}{FI \times Gg \times S.\text{inclin.}}$; that is, as $\frac{\text{Cof. incl.}}{FI \times Gg \times S.\text{inclin.}}$ or as $\frac{1}{FI \times Gg \times \text{tan. inclin.}}$. For nI is as the coſine of inclination, being the ſpace deſcribed upon the inclined plane nI.

Cor. 2. *Taking away all impediments, the circle always keeps the ſame inclination to the horizon.*

For the poſition of the plane FnI is ſuch, that the axes CP, Cq, are both parallel to it. If we ſuppoſe gravity to act by a ſingle impulſe at O, then F will move to n, and P to q. And the plane of the circle endeavouring to deſcend a little at D, and riſe at E; a new point of the circle as t, lying beyond G will inſtantly touch the plane; by which means it leaves the line AG. And ſince at every

every point of contact as G, the pole P moves (at each impulse of gravity) in a line perp. to CG, and also parallel to the tangent arch at G; and the like at every new point of contact; it is plain CP is always alike inclined to the horizon. Consequently, when gravity is continual, the circle coming continually to new points of contact, the axis CP will always revolve round at the same inclination, and therefore the plane of the circle will also have the same inclination to the horizon. Fig. 36.

This might also be proved after the manner of the XXXth Prop. not considering the progressive motion of the circle.

All this is supposing there is no resistance, friction, or other irregularity. But since in fact, the resistance of the air continually lessens its motion, and the smoothness of the plane it runs on, causes the foot or bottom of the circle to slide outward, which continually lessens the inclination, and brings the axis more upright; and the more oblique the plane of the circle is, the faster it slides out. Upon these accounts it can never describe a circle, but only a sort of spiral line; and the plane of the circle descending lower and lower, at last falls flat upon the horizon.

Cor. 3. *Hence a circle moving without any resistance, &c. upon a horizontal plane; will describe a circle upon that plane.*

For the velocity and inclination continuing the same; the curvature of the track described, will be every where alike.

Cor. 4. *And to find the diameter of the circle or orbit described.*

Let t be a very small part of time wherein Gg is described, v the velocity of projection per second, h the space descended by gravity in time t, s and c the sine and cosine of the circles inclination, f =

Fig. 36. $16\frac{1}{12}$ feet; then will ch = ſpace deſcended along the inclined plane Ff; and by the laws of motion,

$$1'' : v :: t'' : Gg = tv.$$

$$\text{and } tt : h :: 1'' : f = \frac{h}{tt}.$$

Then whilſt O has moved through the length Gg, t or I has deſcended through the ſpace ch on an inclined plane parallel to Ff. But we proved $gt = \frac{nI}{s} = \frac{tr}{s} = \frac{ch}{s}$. And therefore the diameter of the orbit $= \frac{\overline{Gg}^2}{tg} = \frac{\overline{Gg}^2 \times s}{ch} = \frac{ttvvs}{ch} = \frac{vvs}{cf} = \frac{vv}{f} \times$ tan. inclination.

Cor. 5. *Alſo to find the periodic time, or time of one revolution.*

Let D = diameter of the orbit, then by Cor. laſt, $\frac{\overline{Gg}^2 \times s}{ch} = D$, and $Gg = \sqrt{\frac{chD}{s}}$. The circumference of the orbit is $\frac{\pi \times \overline{Gg}^2 \times s}{ch}$ (putting $\pi = 3.1416$); and by uniform motion, $Gg : t :: \frac{\pi \times \overline{Gg}^2 \times s}{ch} : \frac{\pi ts \times Gg}{ch}$ the periodic time $= \frac{\pi st}{ch} \sqrt{\frac{chD}{s}} = \pi \sqrt{\frac{ttsD}{hc}} = \pi \sqrt{\frac{SD}{fc}}$.

Or thus,

$Gg\ (tv) : t :$: circumference $\frac{\pi vvs}{fc}$: periodic time $= \frac{\pi vs}{fc} = \frac{\pi v}{f} \times$ tan. inclination.

PROP.

PROP. XXXVI. *Prob.*

Fig. 37.

If DEF *be the ſurface of a right cone, whoſe axis* AE *is perpendicular to the horizon, and* DHFG *a circular plane parallel to the horizon; and if a circle ab revolves round in the periphery* DHFG, *with its axis* 1B2 *always parallel to the ſide of the cone* DE, *where it then is. To find the periodic time of ab, in the circle* DHFG.

Draw DBA perp. to DE, and BC perp. to AC. Let $\pi = 3.1416$, h = ſpace deſcended by gravity in the time t.

Then if the force of gravity be repreſented by AC, the centrifugal force at B to keep the circle *ab* in that poſition, through its whole revolution, will be denoted by BC. Then it will be AC (the gravity) : BC (the cent. force) : : $h : \frac{BC}{AC} h$ = ſpace deſcended towards C, by the force in direction BC = the effect of the centrifugal force. Therefore (by Prop. II.) the periodic time is

$$\pi t \sqrt{\frac{2BC}{\frac{BC}{AC} \times h}} = \pi t \sqrt{\frac{2AC}{h}}.$$

Cor. 1. *Hence the periodic time* $= \pi t \sqrt{\frac{2BC}{h} \times}$ *tan. inclination* ABC.

For S.A : S.B : : BC : AC $= \frac{S.B}{S.A} \times BC = \frac{S.B}{cos.B} \times BC =$ tan. B $\times$ BC; whence, the periodic time $= \pi t \sqrt{\frac{2BC}{h} \times}$ tan. ABC.

Cor. 2. *Draw* BI *parallel to* DE, *and* DL *parallel to* BC; *then the periodic time* $= \frac{\pi t \times BC \times \sqrt{2}}{\sqrt{CI} \times h}$.

For $AC = \frac{BC^2}{CI}$.

Cor.

Fig. 37. Cor. 3. *Hence the curve deſcribed on the conic ſurface, is the ſame as that deſcribed on a horizontal plane, as explained in the laſt Prop. All the difference is, that the conic ſurface hinders the circle ab from ſliding ontward, which the horizontal plane cannot do; except it be ſuppoſed to be ſo rough, that the circle cannot ſlide on it.*

36. For in Cor. 4. of the laſt Prop. v is put for the velocity of G along AB; but putting it for the velocity at C, you will have the diameter of the orbit paſſing through C: that is, (fig. 37.) inſtead of 2DL
37. we ſhould find 2BC for the diameter D of the orbit. And by Cor. 5, the periodic time is $\pi t\sqrt{\frac{sD}{hc}}$. And in Cor. 1. of this Prop. the periodic time is $\pi t\sqrt{\frac{2BC}{h}}\times$ tan. inclination; which is equal to the former, becauſe $D = 2BC$, and $\frac{s}{c}$ = tan. of the inclination. So the curve is the ſame in both caſes.

PROP. XXXVII. *Prob.*

38. *To explain the motion of a top, or ſuch like whirling body.*

Let ABC be a top whirling about in the order AEC; FDG a circle deſcribed by any point D in the ſurface, K its center of gravity, BKS the axis of the top.

If the top be upright upon the foot B; that is, if BS be perp. to the horizon, and moves ſwiftly about; it will continue upright till the motion ſlacken. But when it is going to fall, it will lean to one ſide; therefore ſuppoſe D to be the loweſt point in the circle FG; then the top endeavours by its gravity to deſcend towards D. Let the force of gravity alone move the point D through the ſpace Do, in a very ſmall time; during which, the rotary

motion

motion would carry the point D to *r*. Compleat the parallelogram D*rpo*, and the point D will be carried through D*p*; that is, the circle FDG will come into the position D*p*; and therefore the axis BKS (perp, to the circle FG) will be moved in a direction towards H, perp. to DK; and the poiht S moved to *n*; the particle S*n* being parallel to D*p*. After the ſame manner by a new impulſe of gravity at *p*, the loweſt point; the circle FDG wiil be moved into a new poſition, below D*p*, and the point S carried from *n* to *t*. And ſo by every impulſe of gravity, the point S will be moved gradually forward, through the circle S*ntqlz*; and thus the top recovers itſelf from falling; the motion of S being always parallel to that of D. And therefore the motion of the axis BS will be the ſame way about, as the top's motion is. And thus the point S will continue to make ſeveral revolutions by a ſlow motion, whilſt the top makes its revolutions about its axis, by a ſwift motion. Fig. 38.

Cor. 1. *This motion of the top and its axis, is ſimilar to the motion of an oblong ſpheroid, and its nodes.*

For (Cor. 6. Prop. XXX.) the nodes move the ſame way about as the body revolves, and ſo does the axis of the top; and therefore this motion may be called the motion of the nodes of the top.

Cor. 2. *When the top's motion is very ſwift, the circle* Sql *is very ſmall; and as it grows ſlower, that circle will grow bigger and bigger, till the top falls.*

For when the top's motion is very ſwift, D*r* will be greater, and the angle *r*D*p* leſs; and the circle D*p* will deviate leſs from DG. And gravity having little power to diſturb its motion, the circle S*ql* will be extremely ſmall, and the top will revolve about the axis in appearance unmoved. But as the top's

Fig. 38. top's motion by reſiſtance and friction grows leſs and leſs, D*r* will be leſs; and the circle D*p* will deviate more from DG; that is, gravity will have more and more power to diſturb its motion; and the axis BS will deſcribe a greater and greater circle with the point S, or rather a ſpiral, till at laſt the top falls down.

39. Cor. 3. *As the top grows ſlow, and the motion weak, and the pole* S *deſcribes greater and greater circles, the foot* B *is thrown out to the oppoſite ſide, deſcribing a circle* B*b*, *which is greater as* S*ql is greater; and goes the ſame way about.*

For the center of gravity K always endeavours to be at reſt, whilſt the body revolves about. Therefore when the top grows weak, and the pole S deſcribes greater circles, the foot B is thrown further out to the oppoſite ſide; and being always oppoſite, will deſcribe a circle proportional to S*ql*; the foot B going the ſame way about as S does. And theſe circles B*b* will continually grow greater and greater till the top falls down. Till then the top rolls about and about from the poſition CAB to the oppoſite poſition *cab*, till the motion end, and the top falls down. And theſe are the principal phœnomena of the motion of a top.

FINIS.

ERRATUM, in Plate, Fig. 23.

P*n* ſhould be perp. to F*p*.

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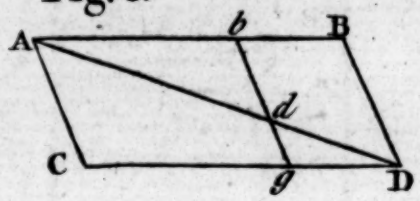
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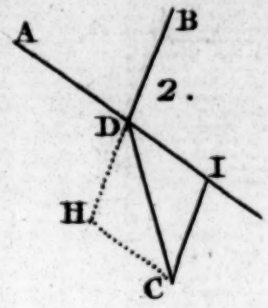
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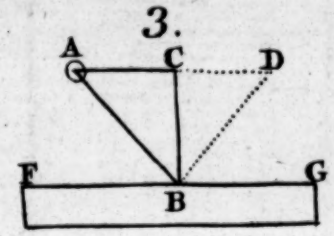
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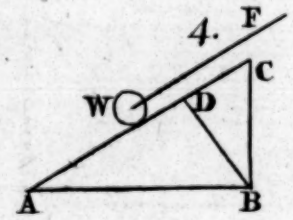
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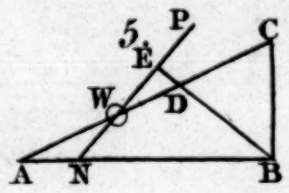
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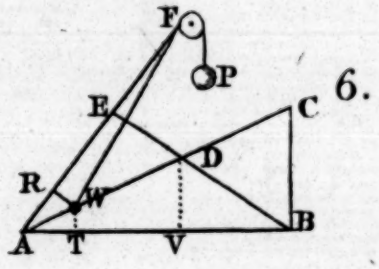
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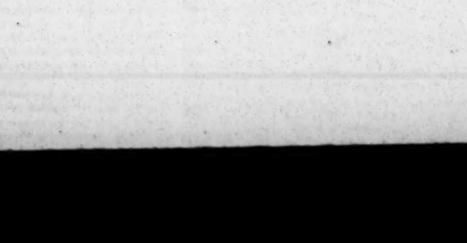
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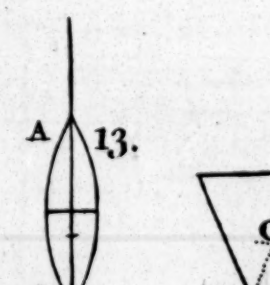
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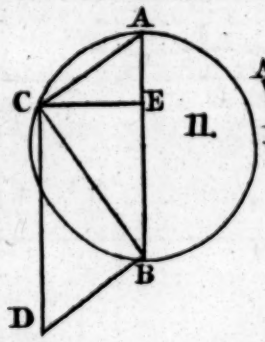
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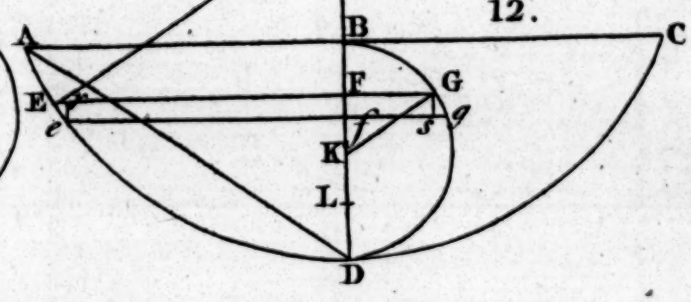
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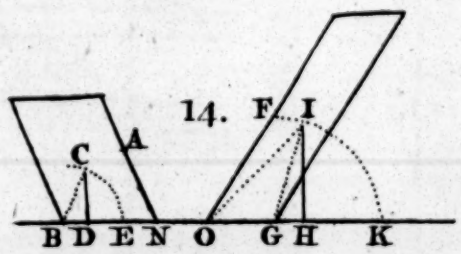
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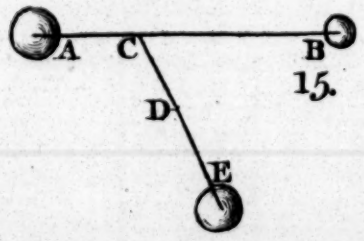
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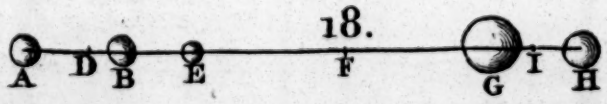
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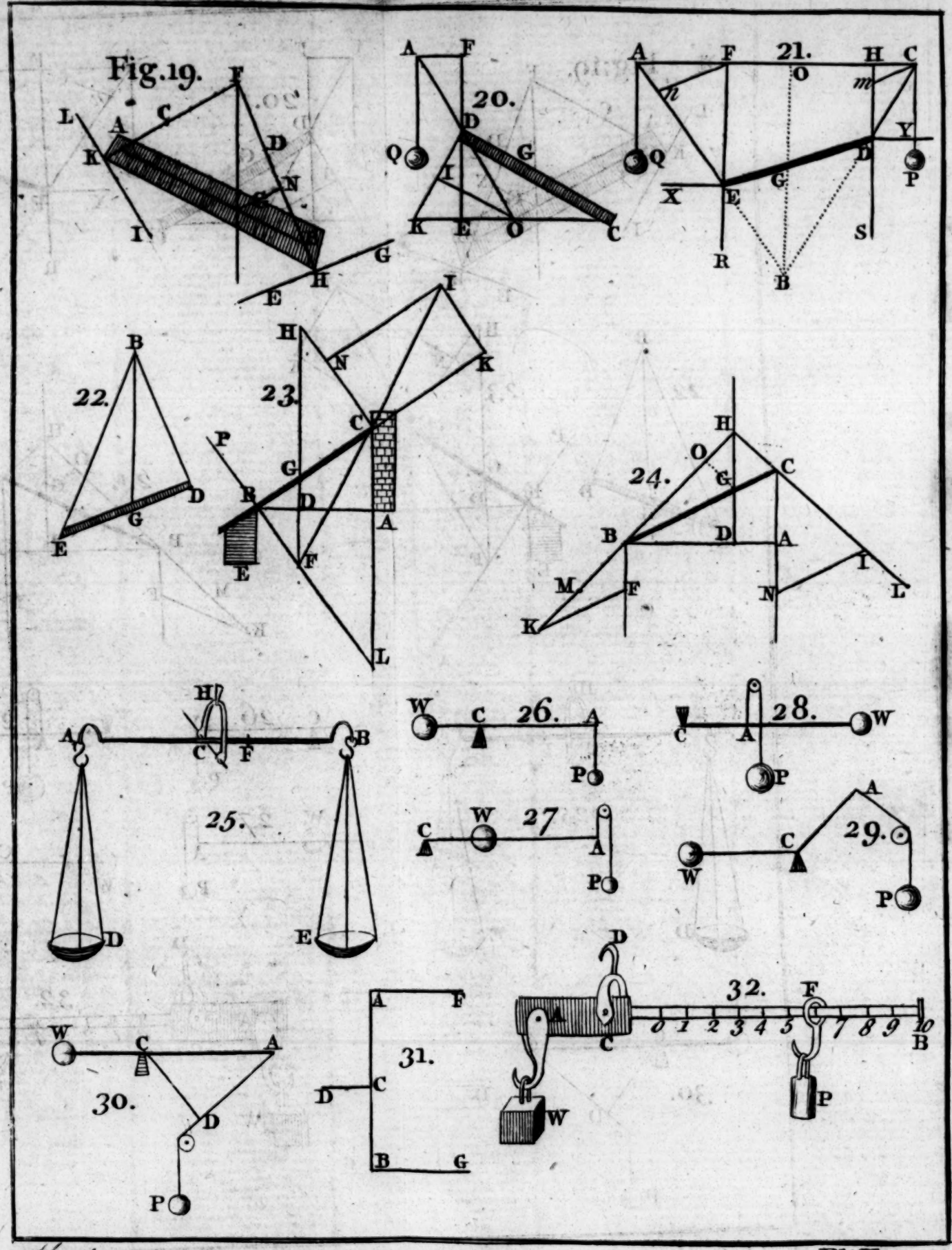

Mechanics

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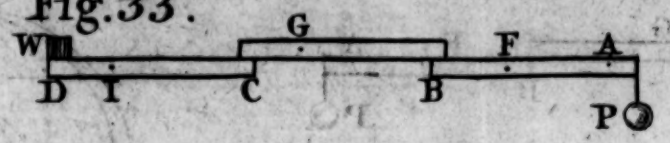
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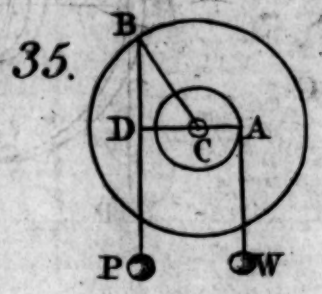
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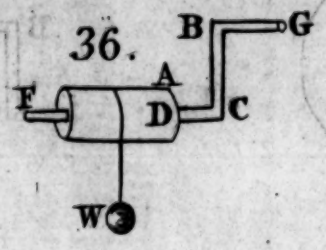
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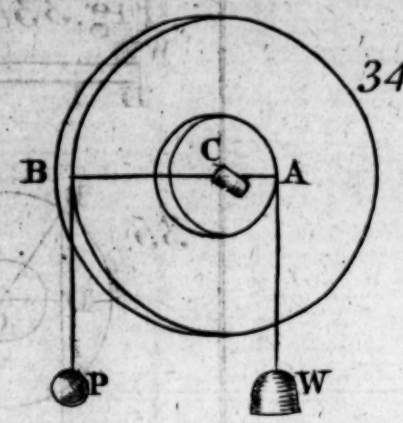
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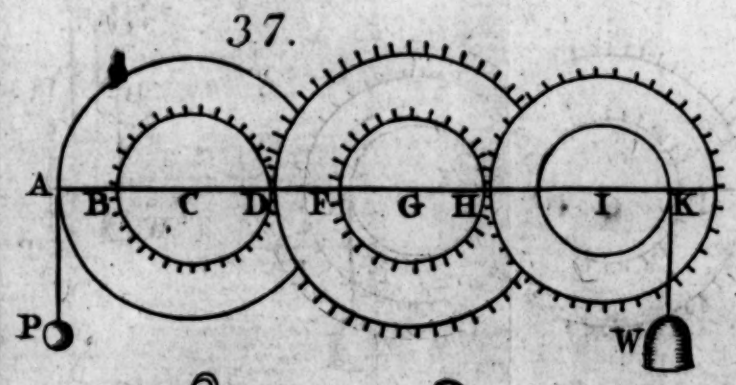
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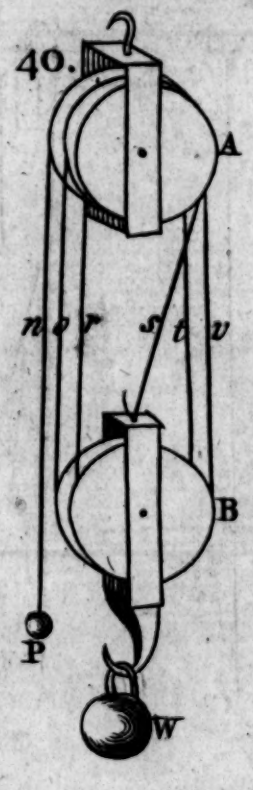
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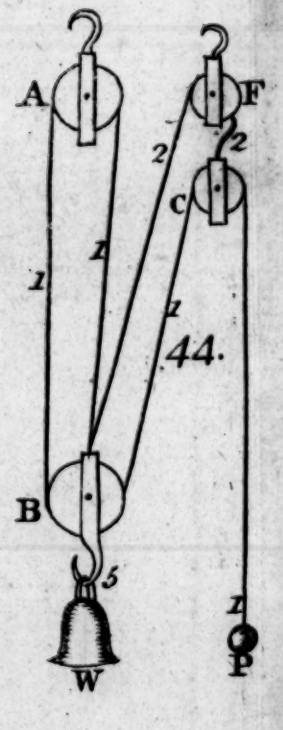
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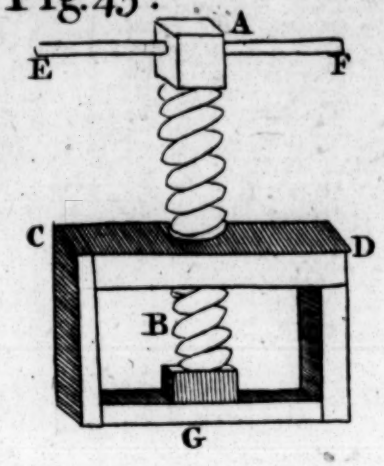

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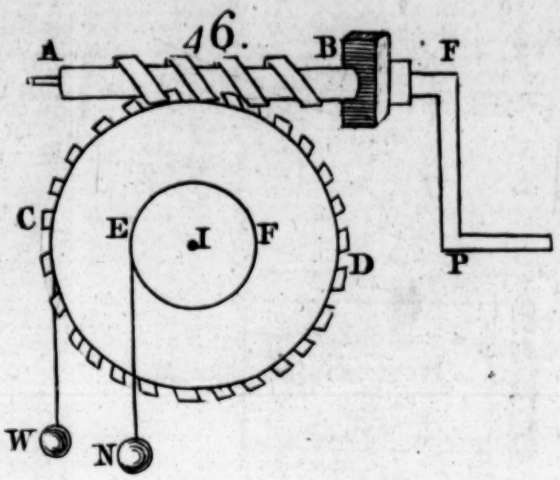

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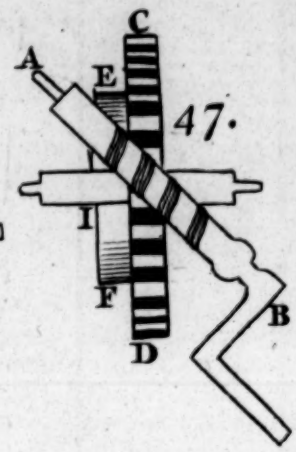

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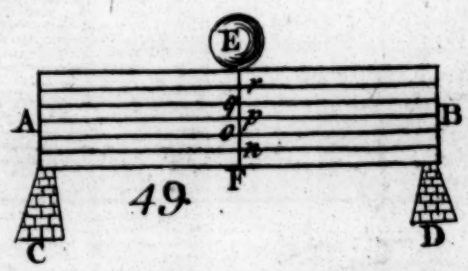

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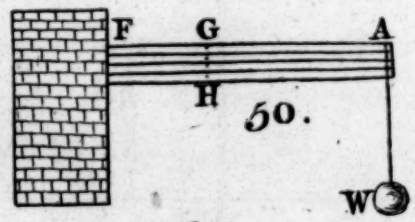

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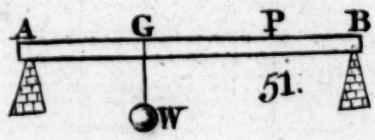

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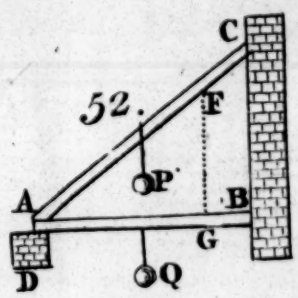

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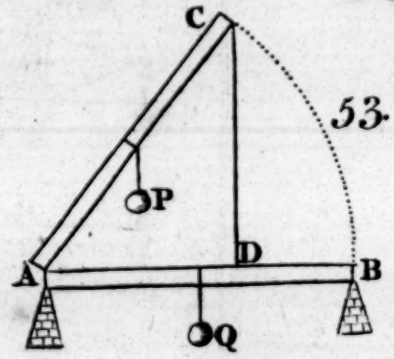

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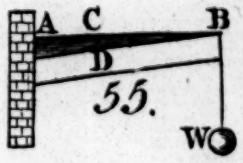

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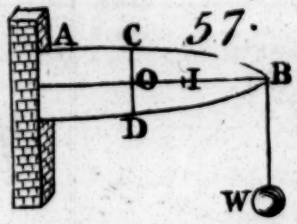

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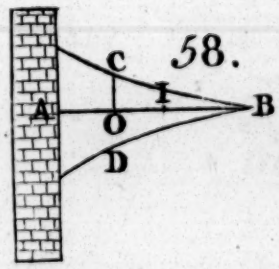

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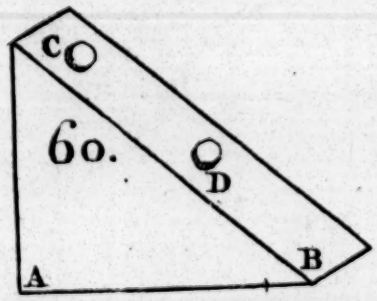
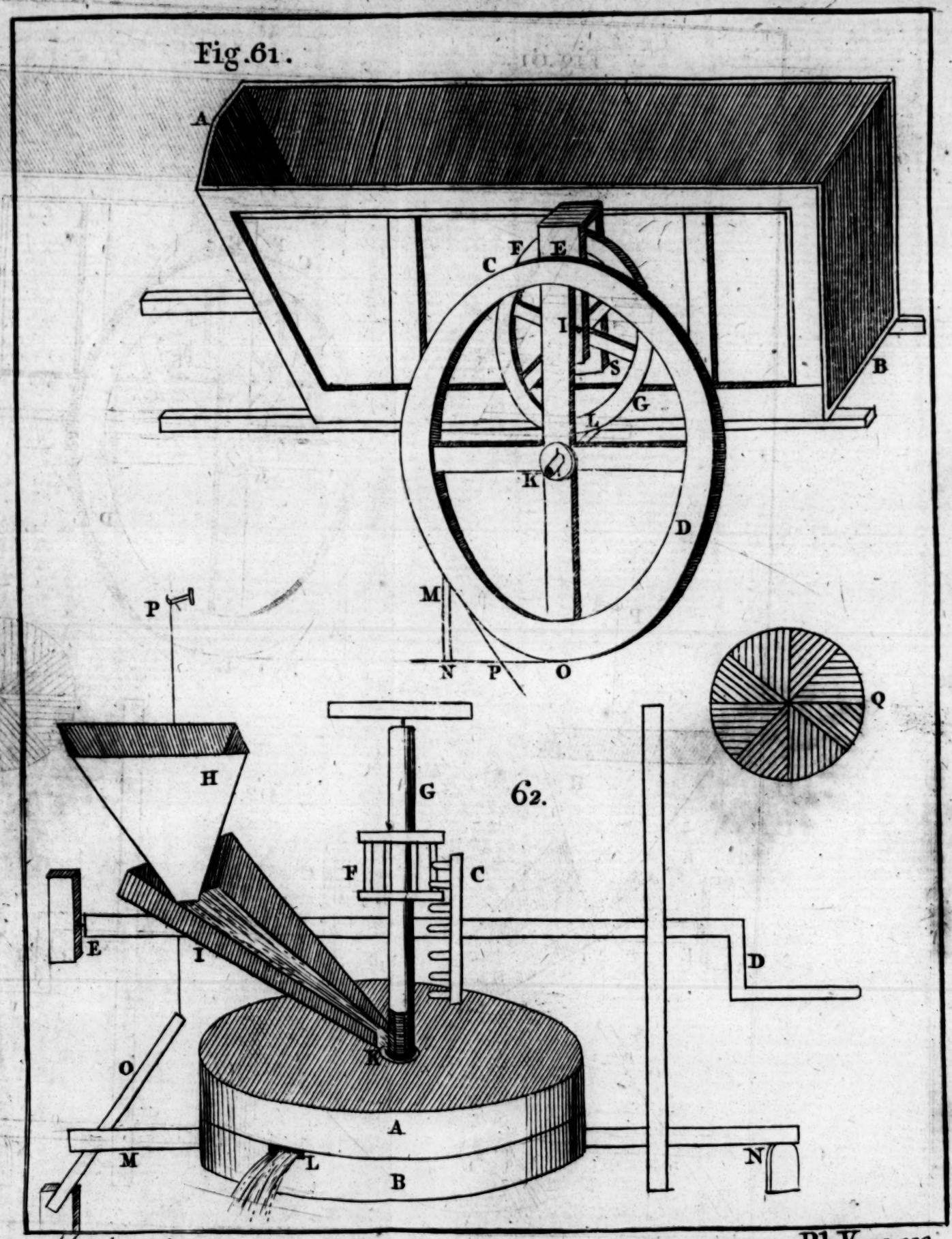


Pl. IV. *pa.108*

Mechanics

Fig.61.



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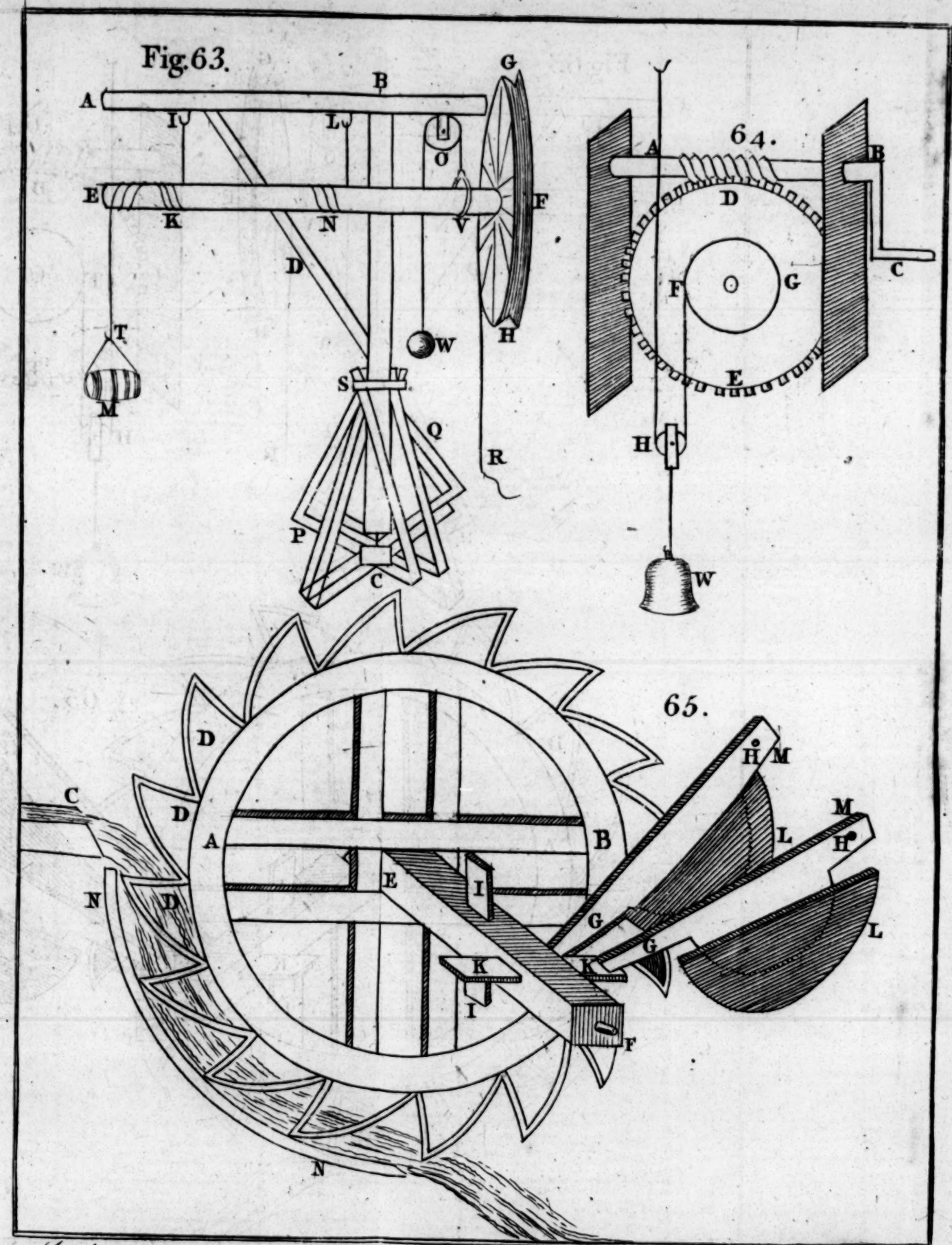
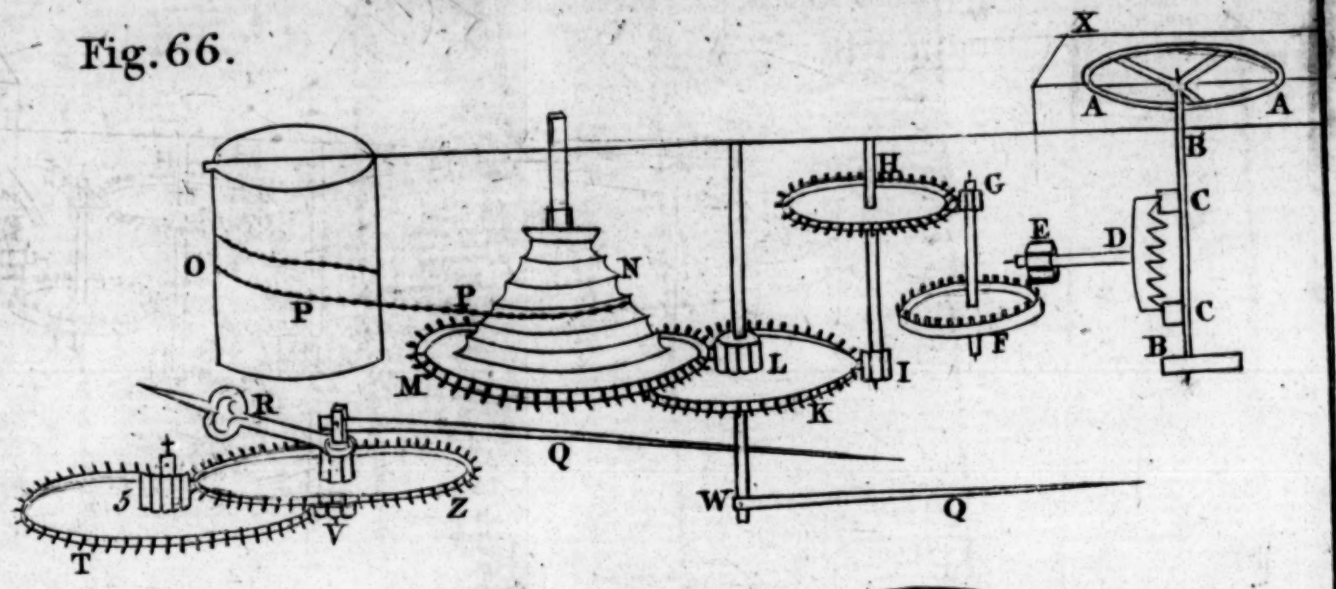


Mechanics

Pl. VI. pa. 114

Fig. 66.



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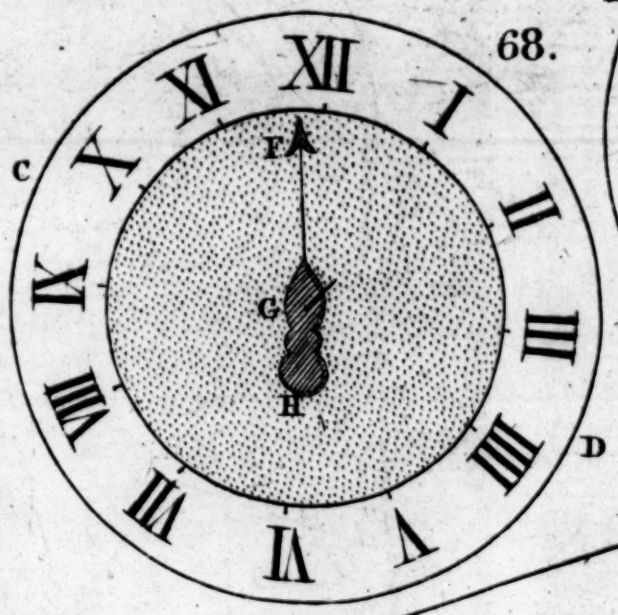
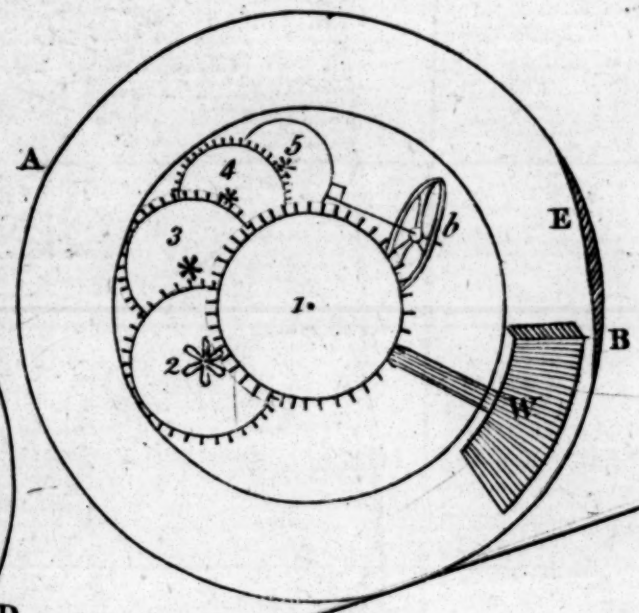
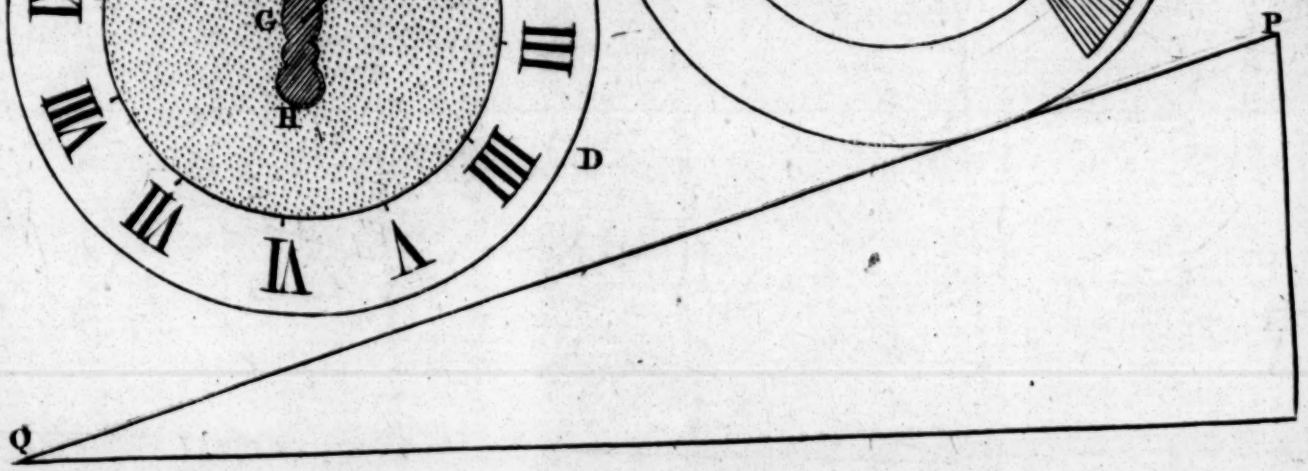
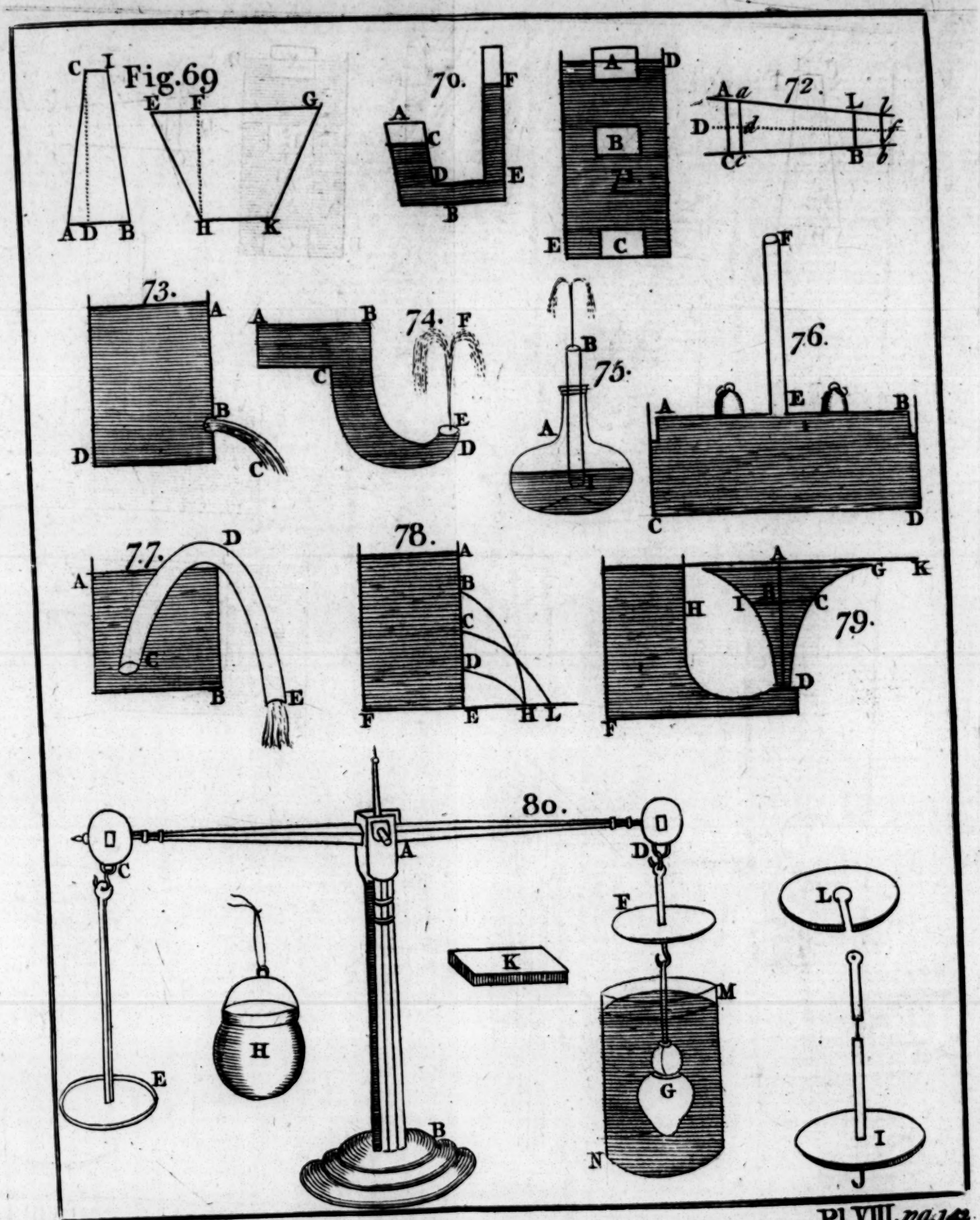






Pl. VII. pa. 122

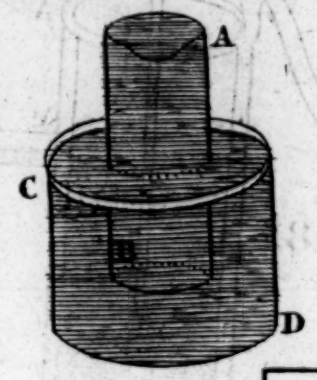
Mechanics



Mechanics

Pl.VIII.pa.142

Fig. 81.



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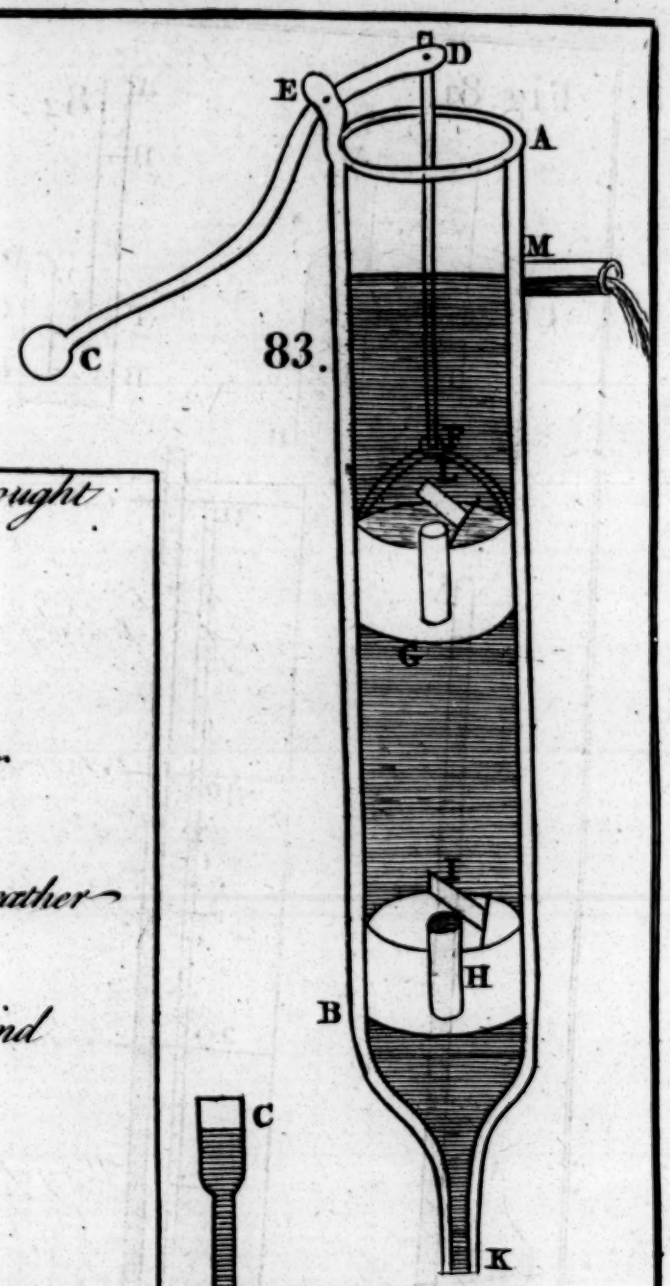

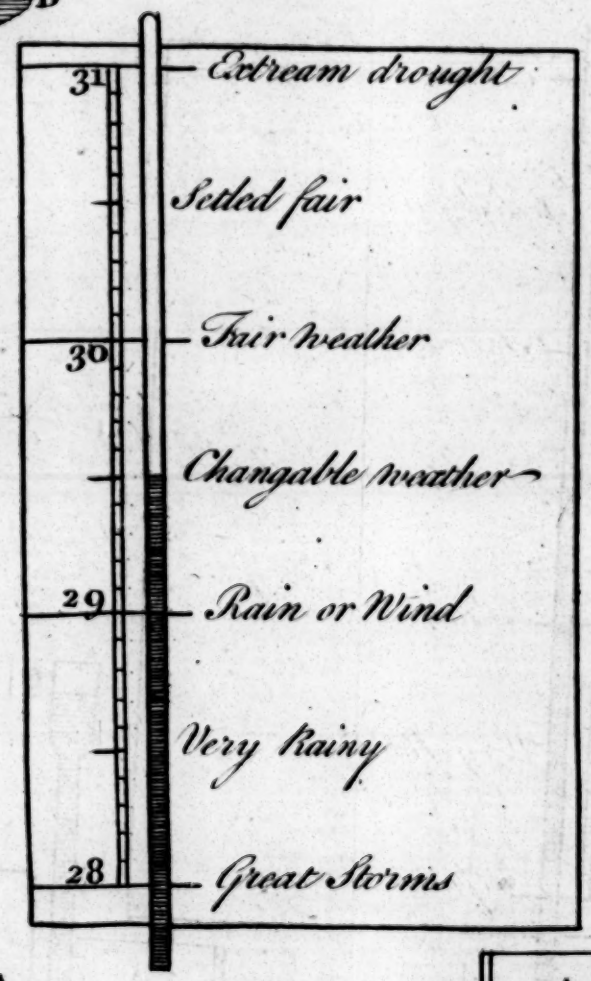

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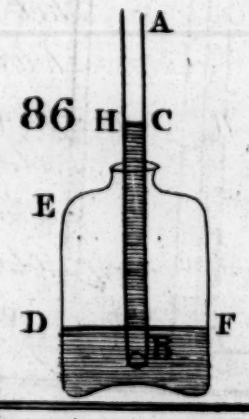

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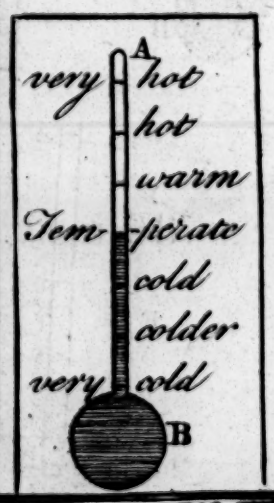


87

Mechanics

Pl. IX. pa 154

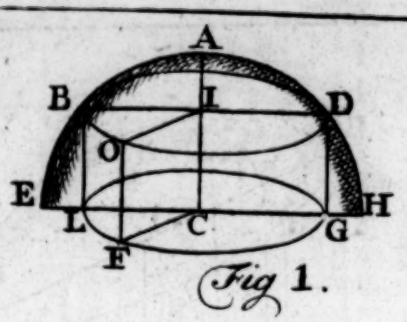
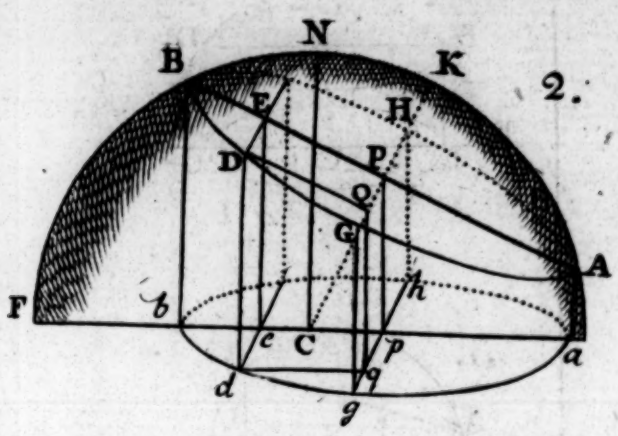
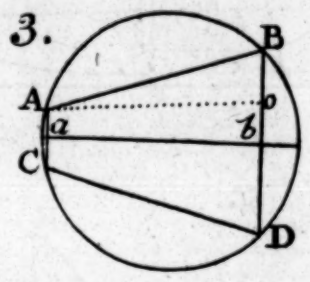
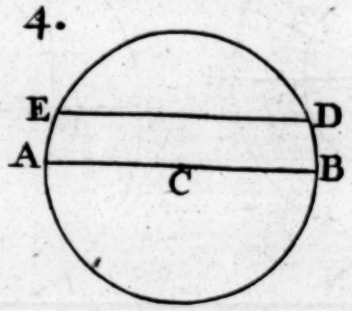
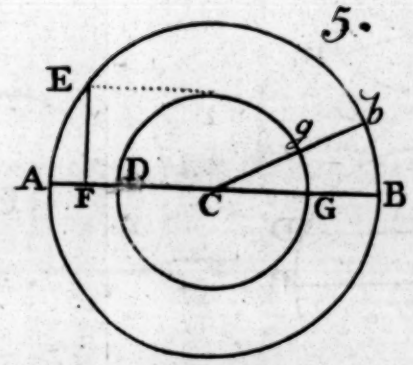


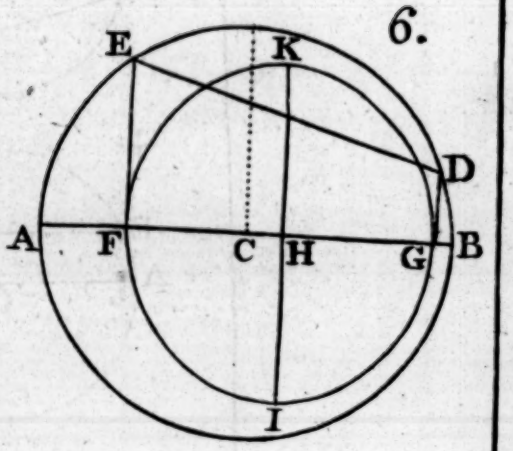
Fig 1.

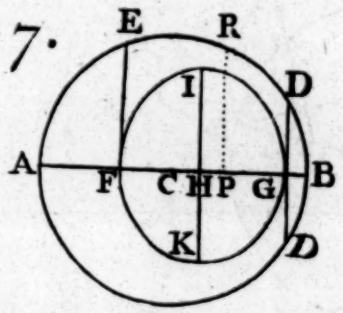


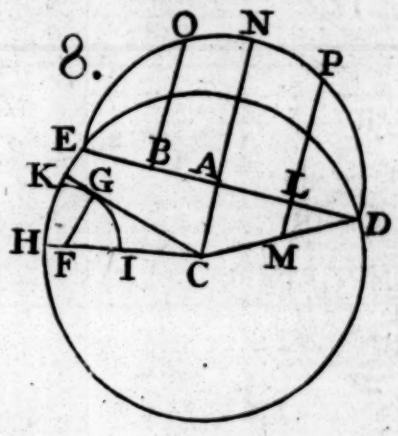


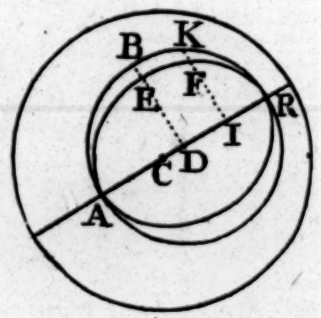


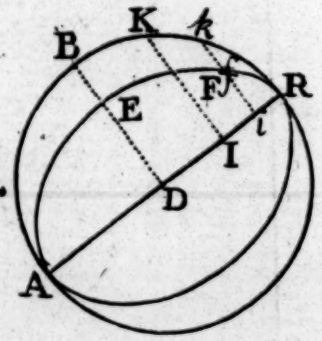






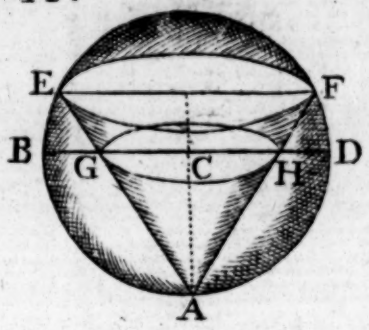






Projection.

11.



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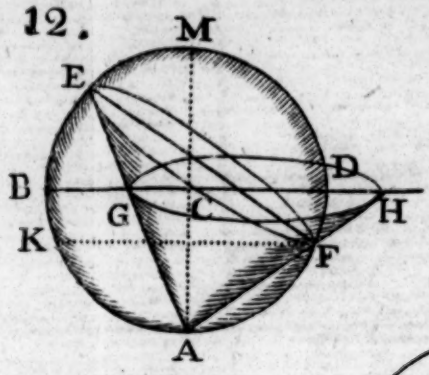

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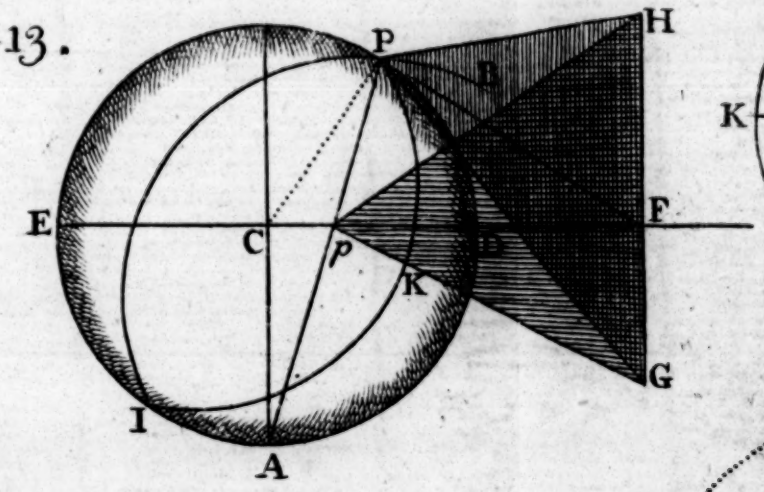

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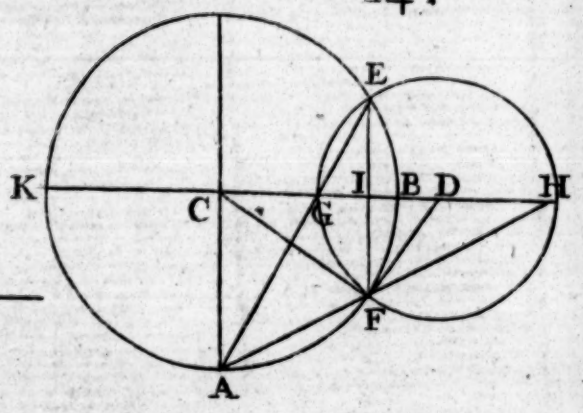

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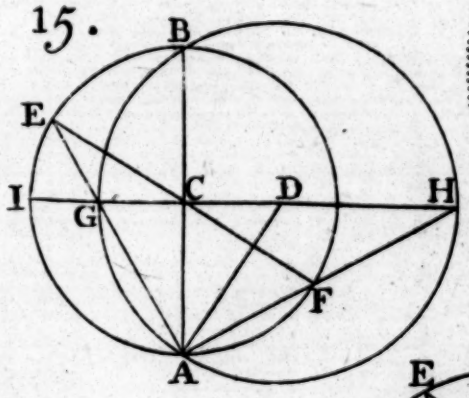

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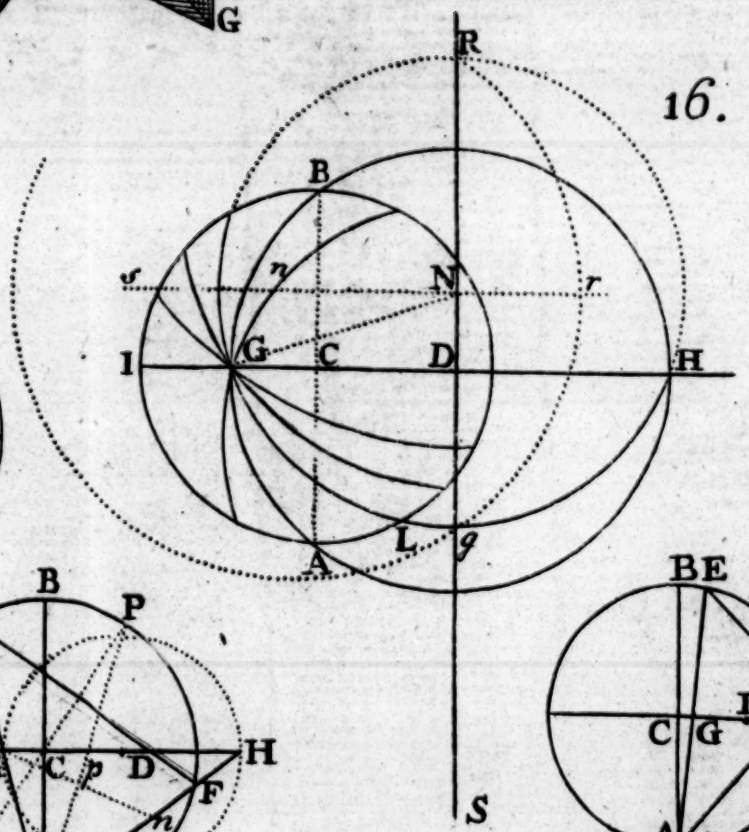

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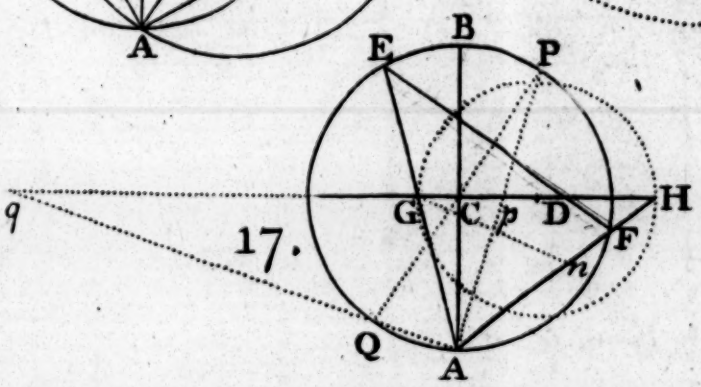

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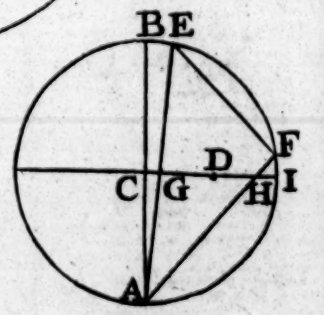

Projection.

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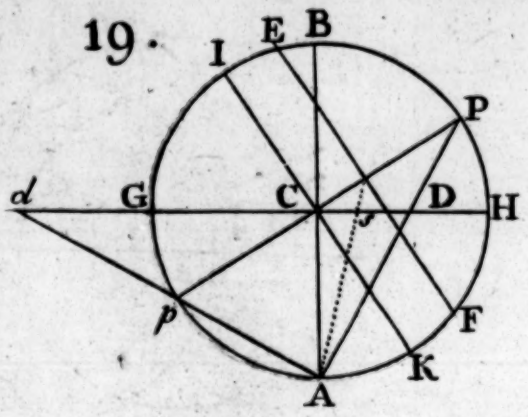

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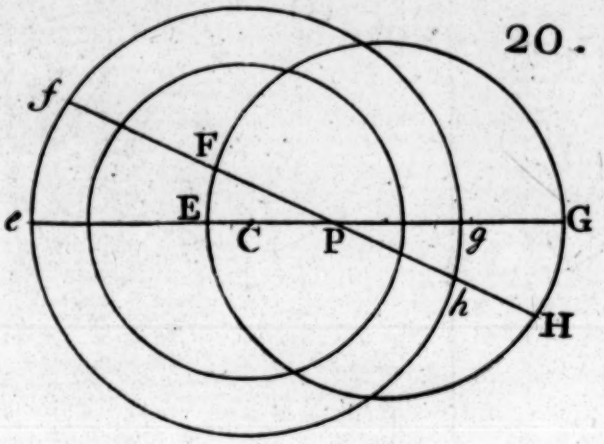

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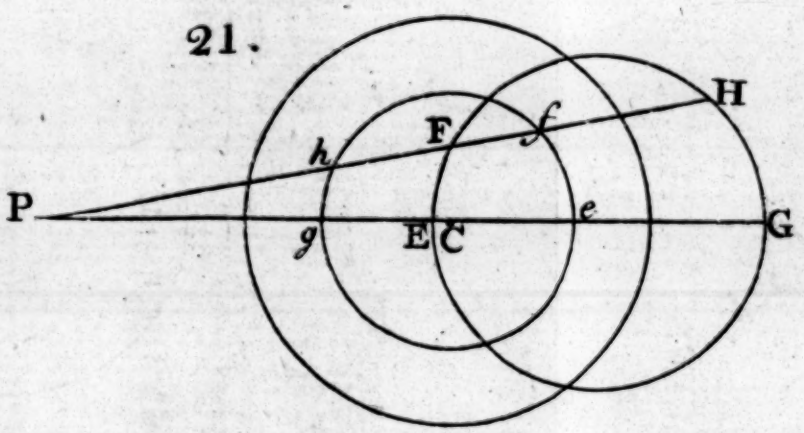

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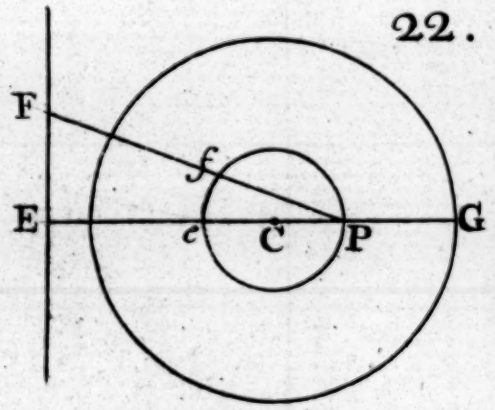

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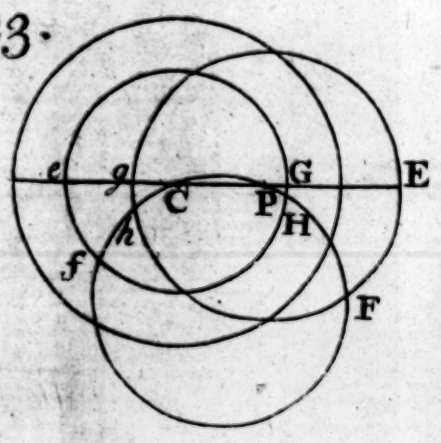

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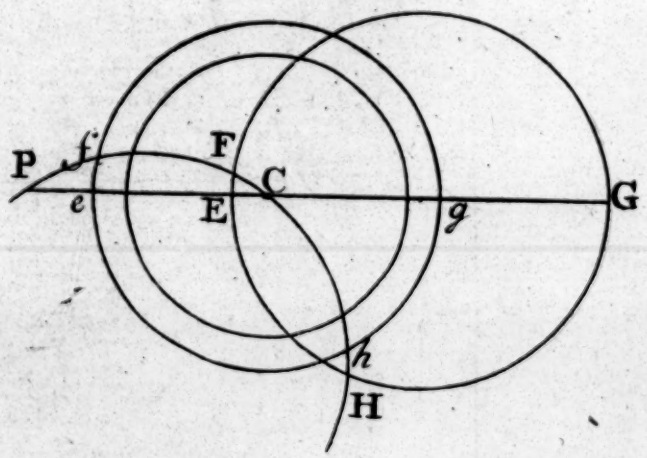

Projection.

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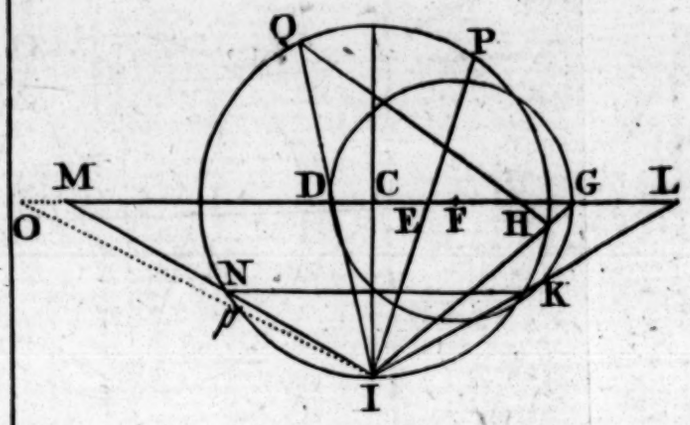

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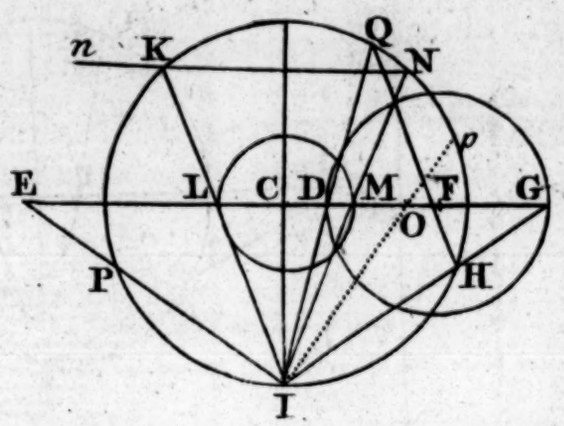

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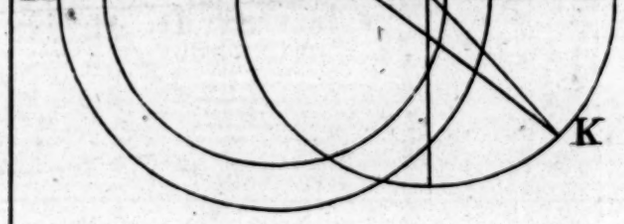

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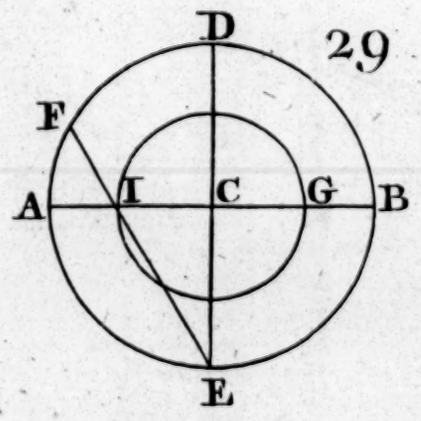

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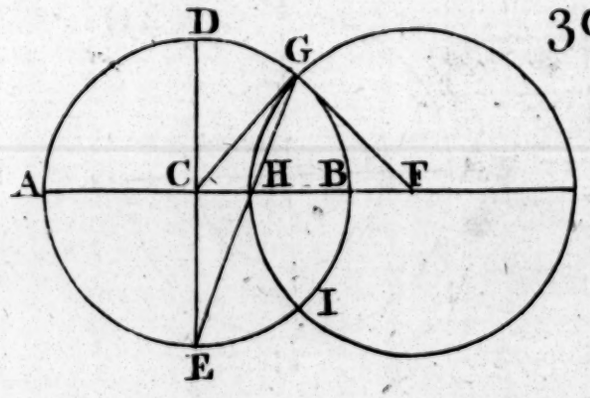

Projection.

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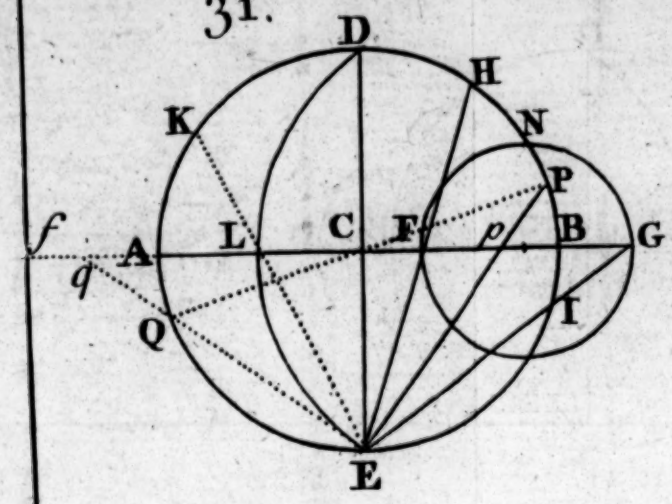

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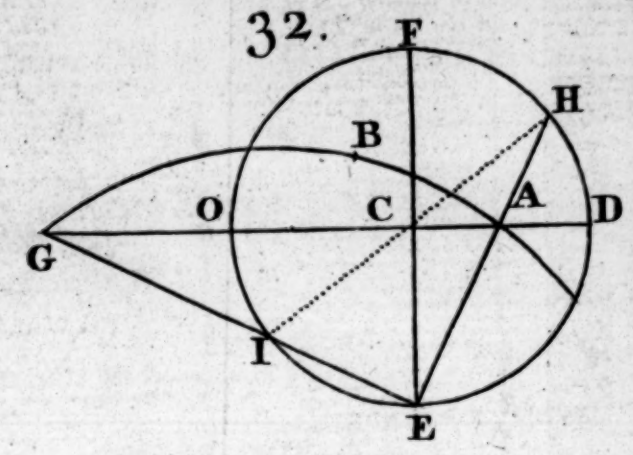

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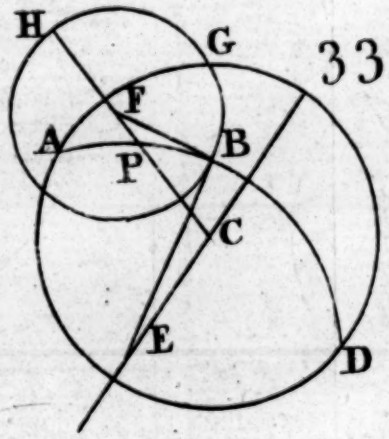

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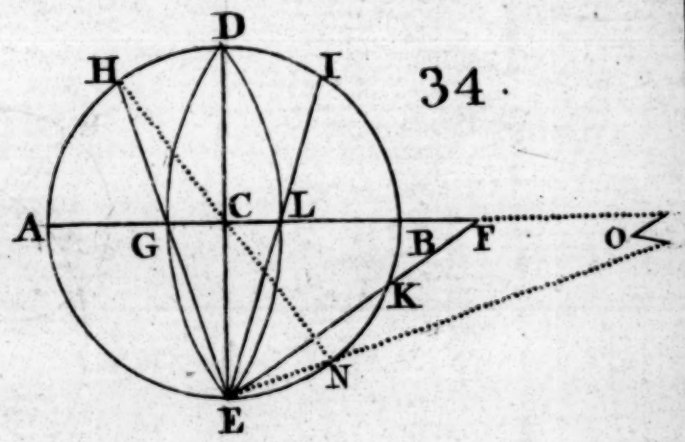

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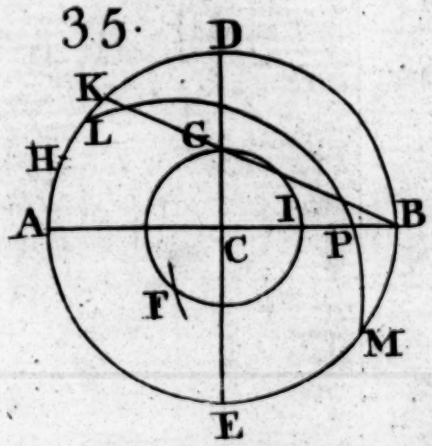

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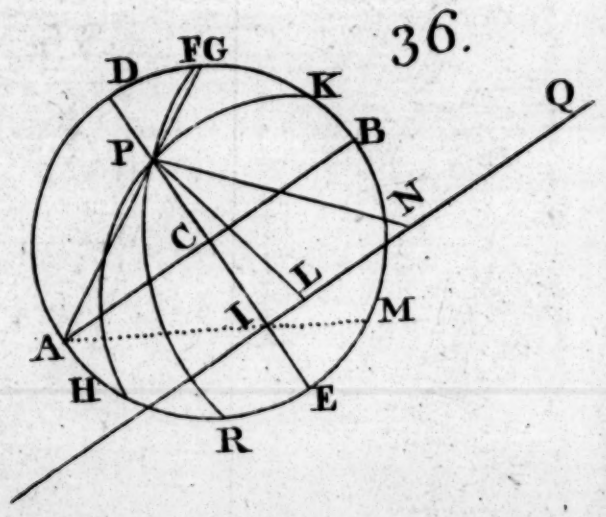

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Projection.

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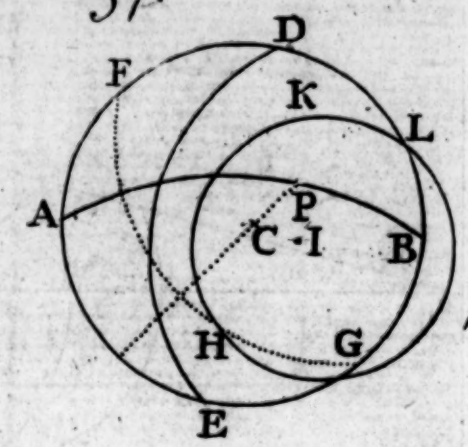

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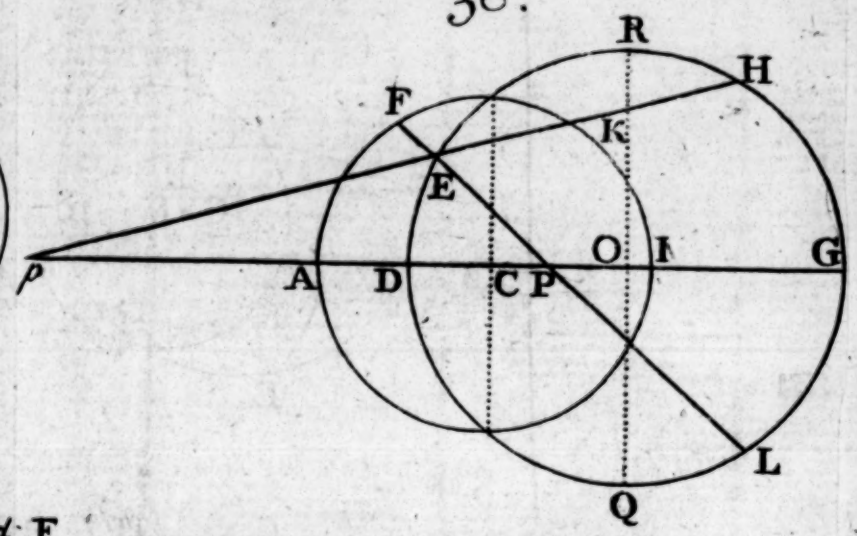

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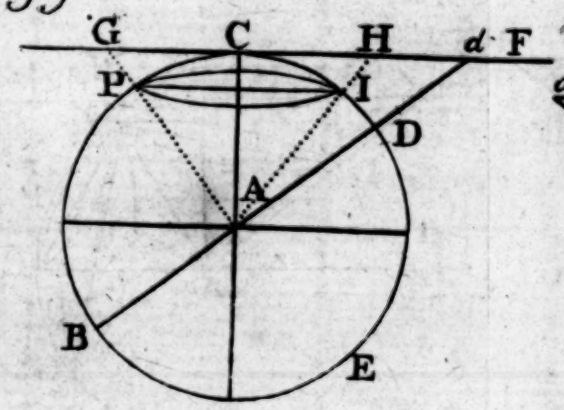

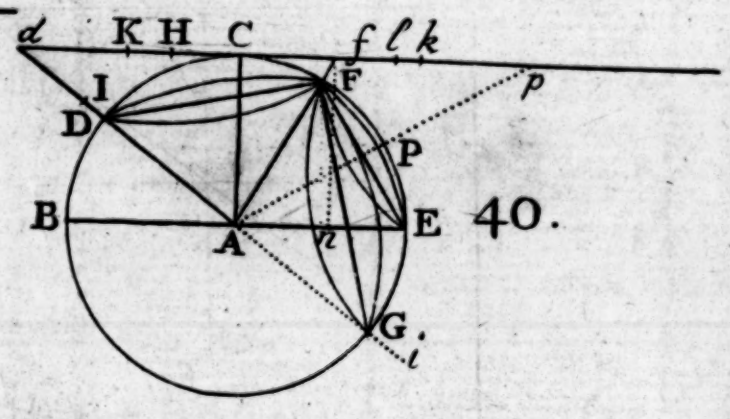

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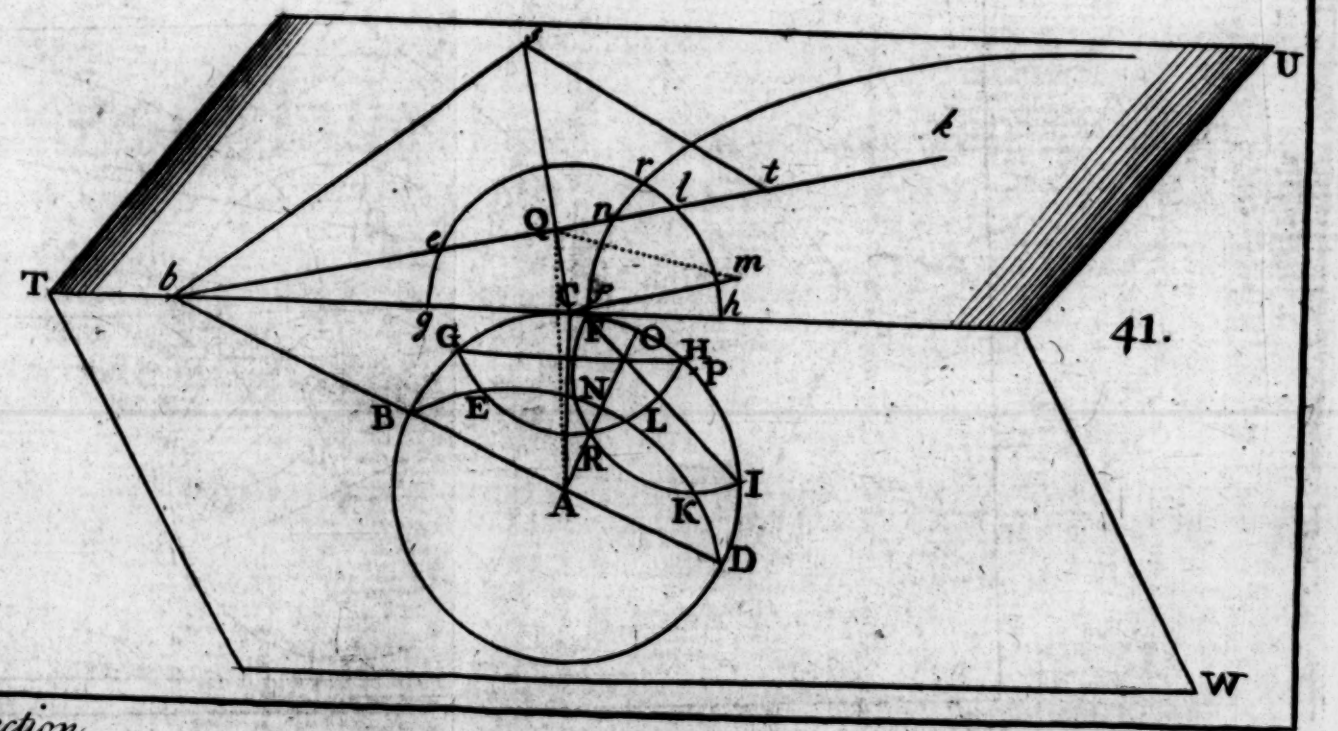
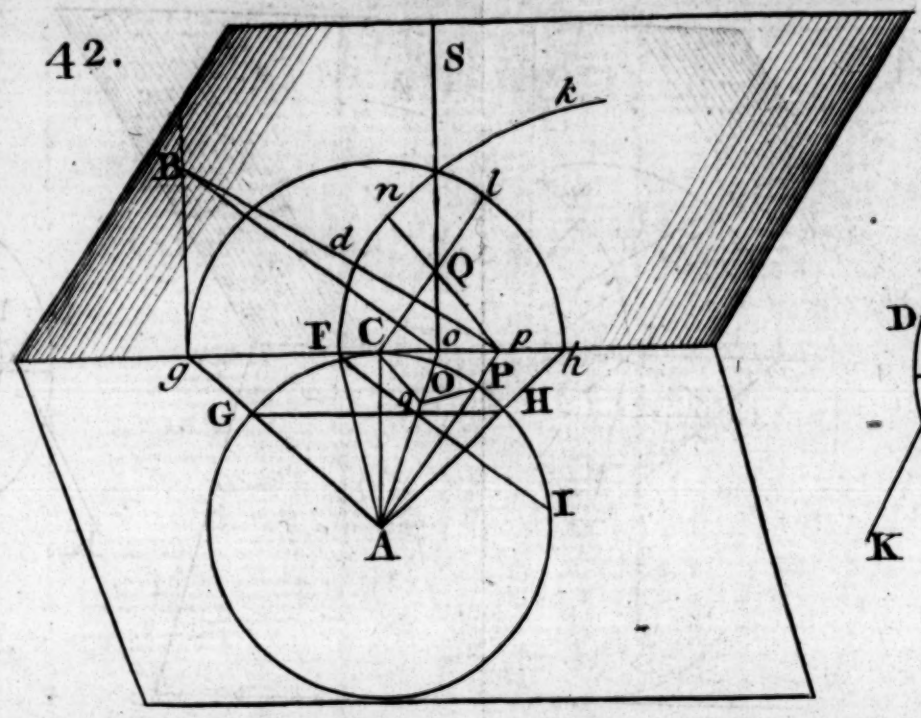


41.

Projection.

42.



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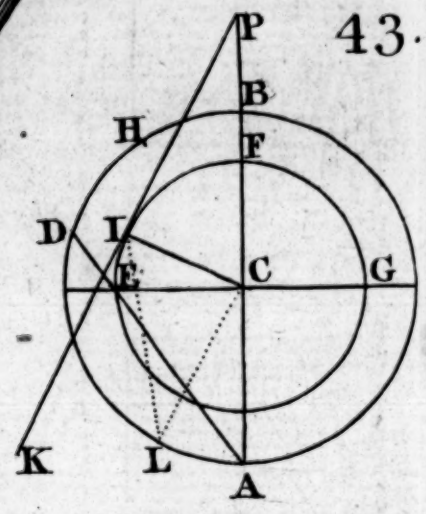
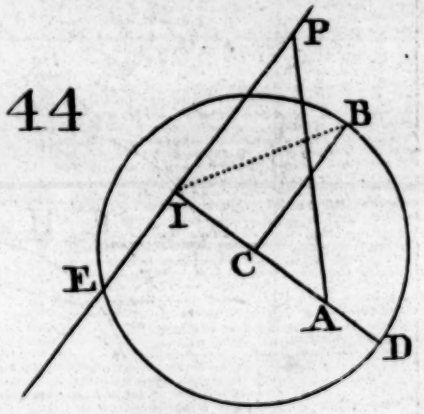


44



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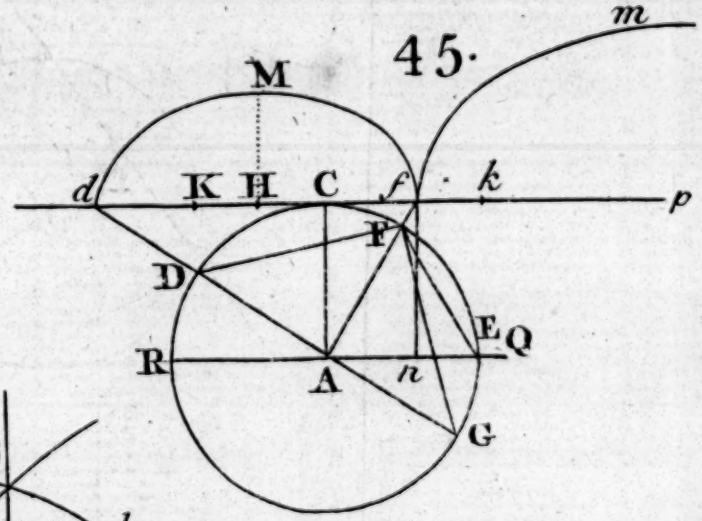

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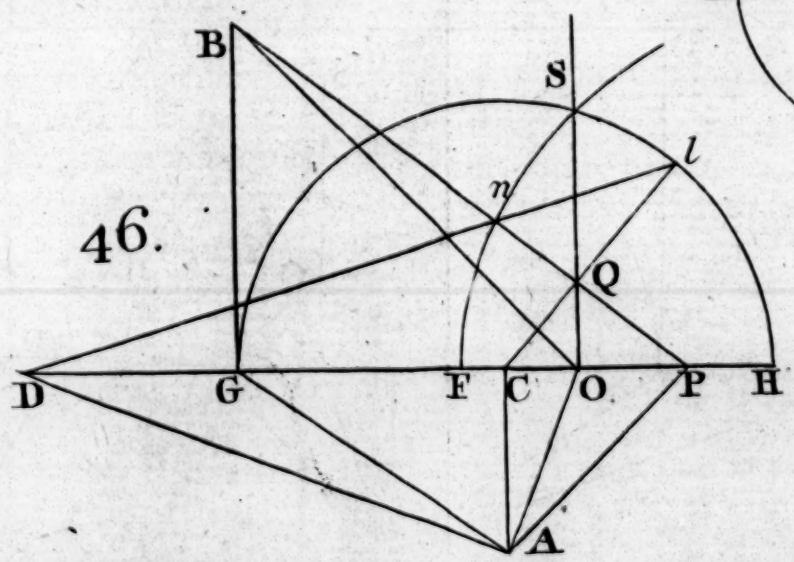

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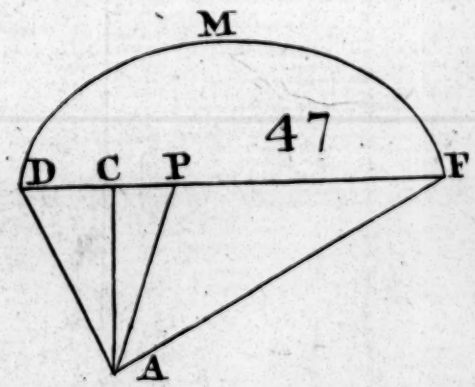
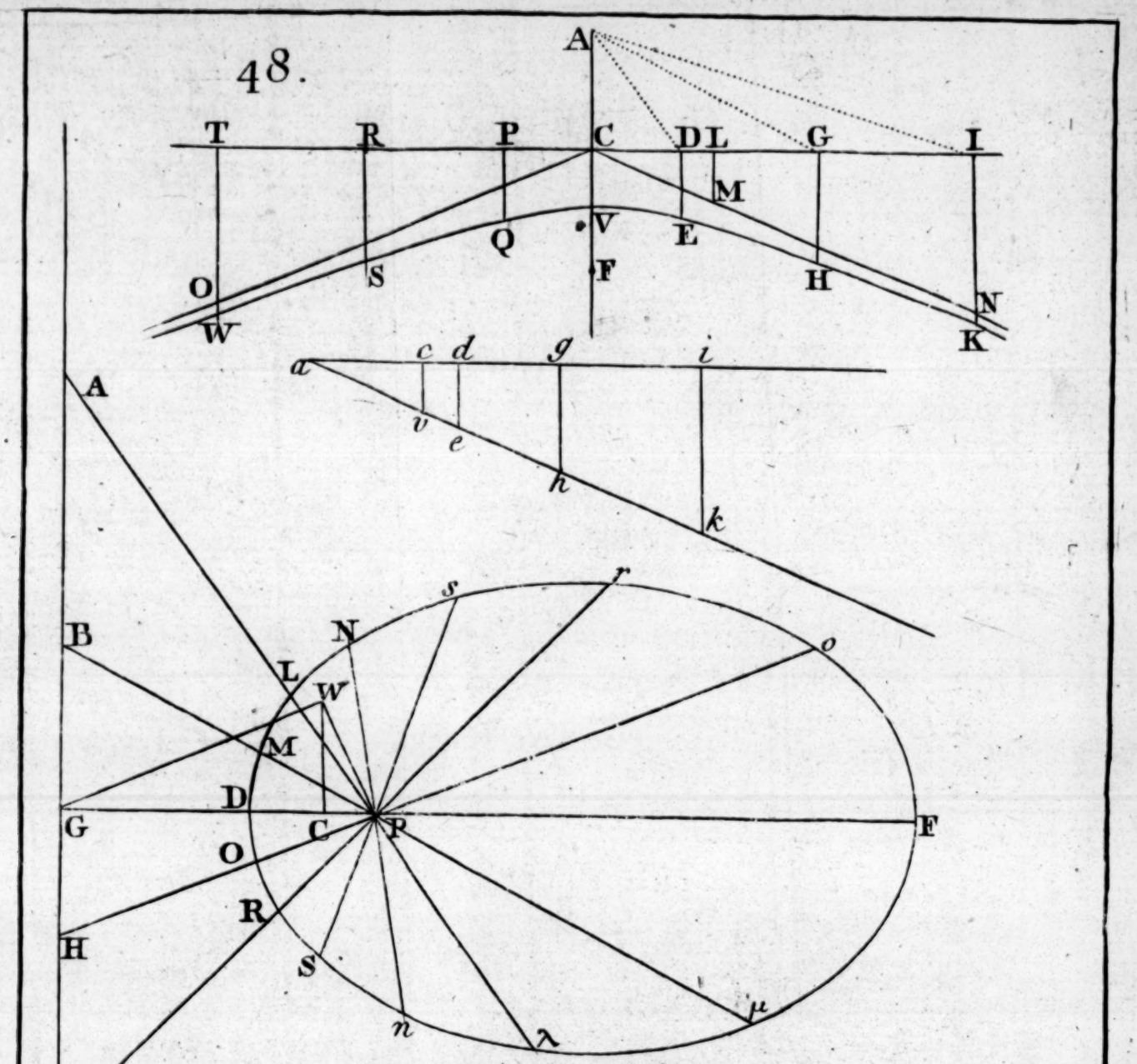


Projection.

VII. pa. 208.

48.



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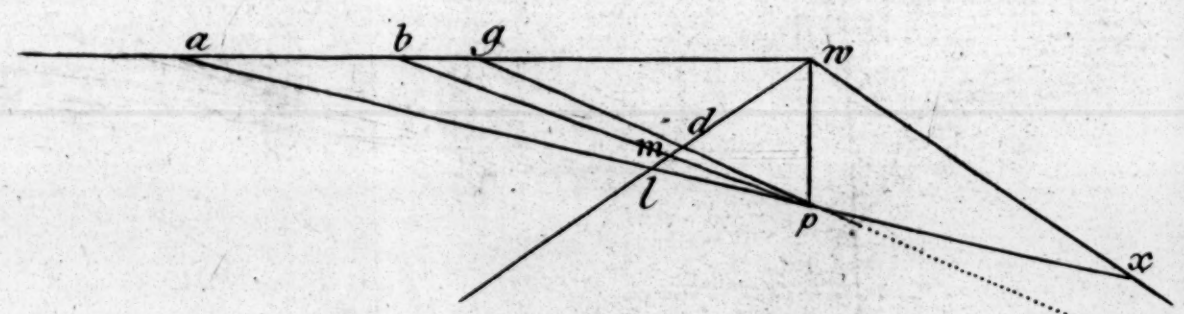
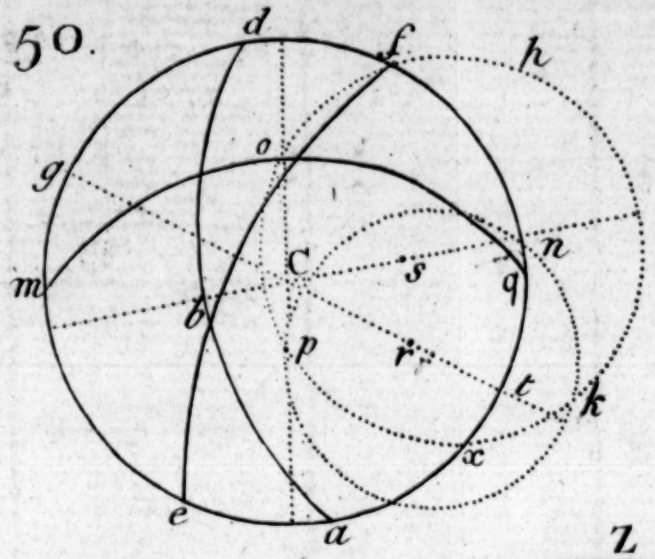


Projection.

VIII. p. 270

50.



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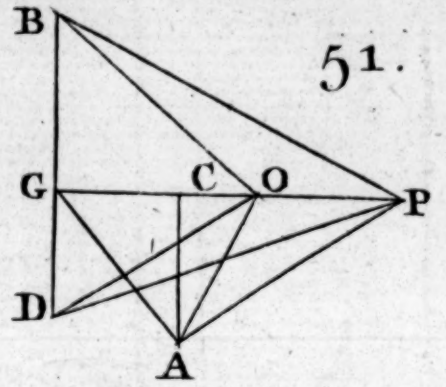

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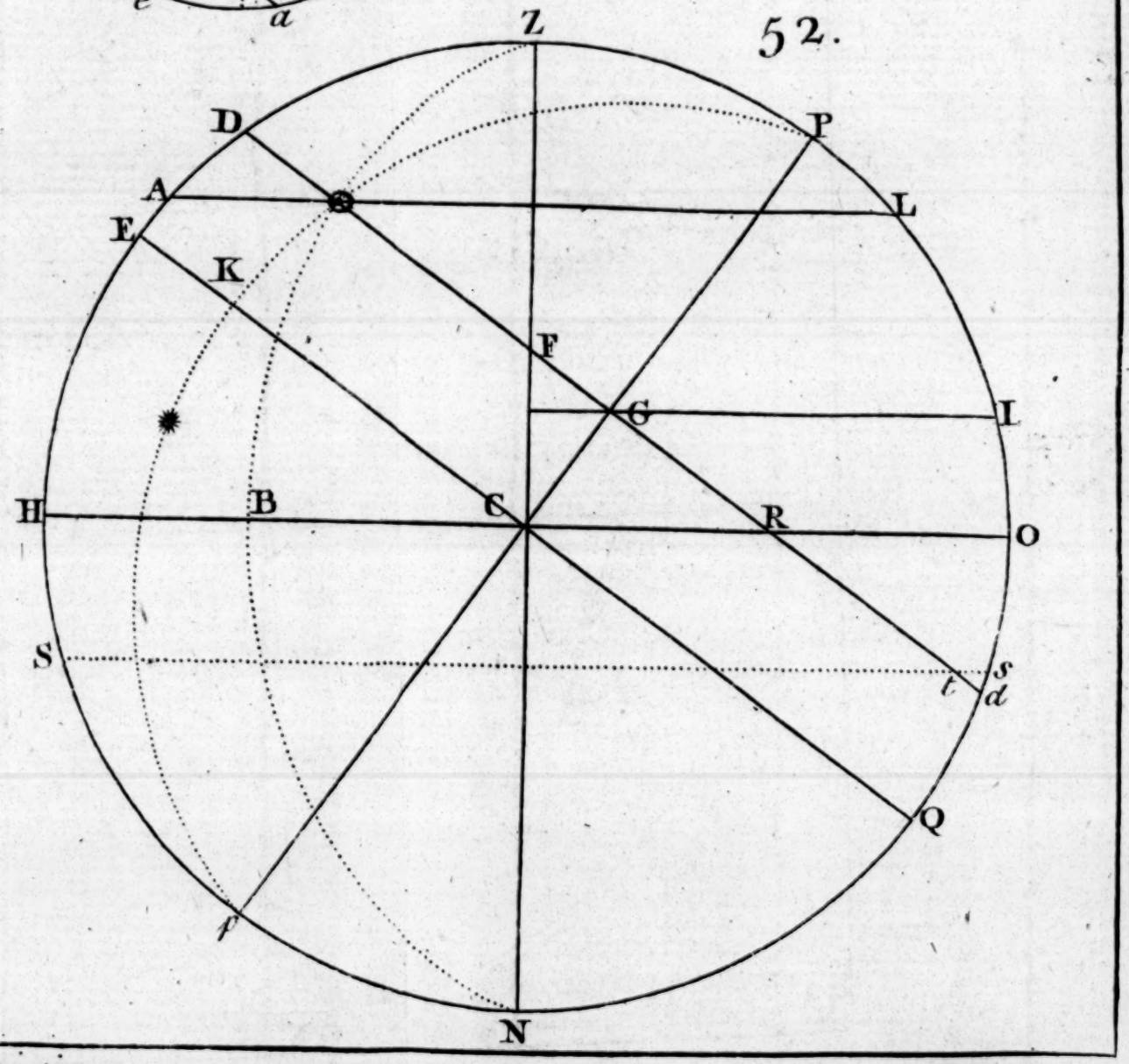
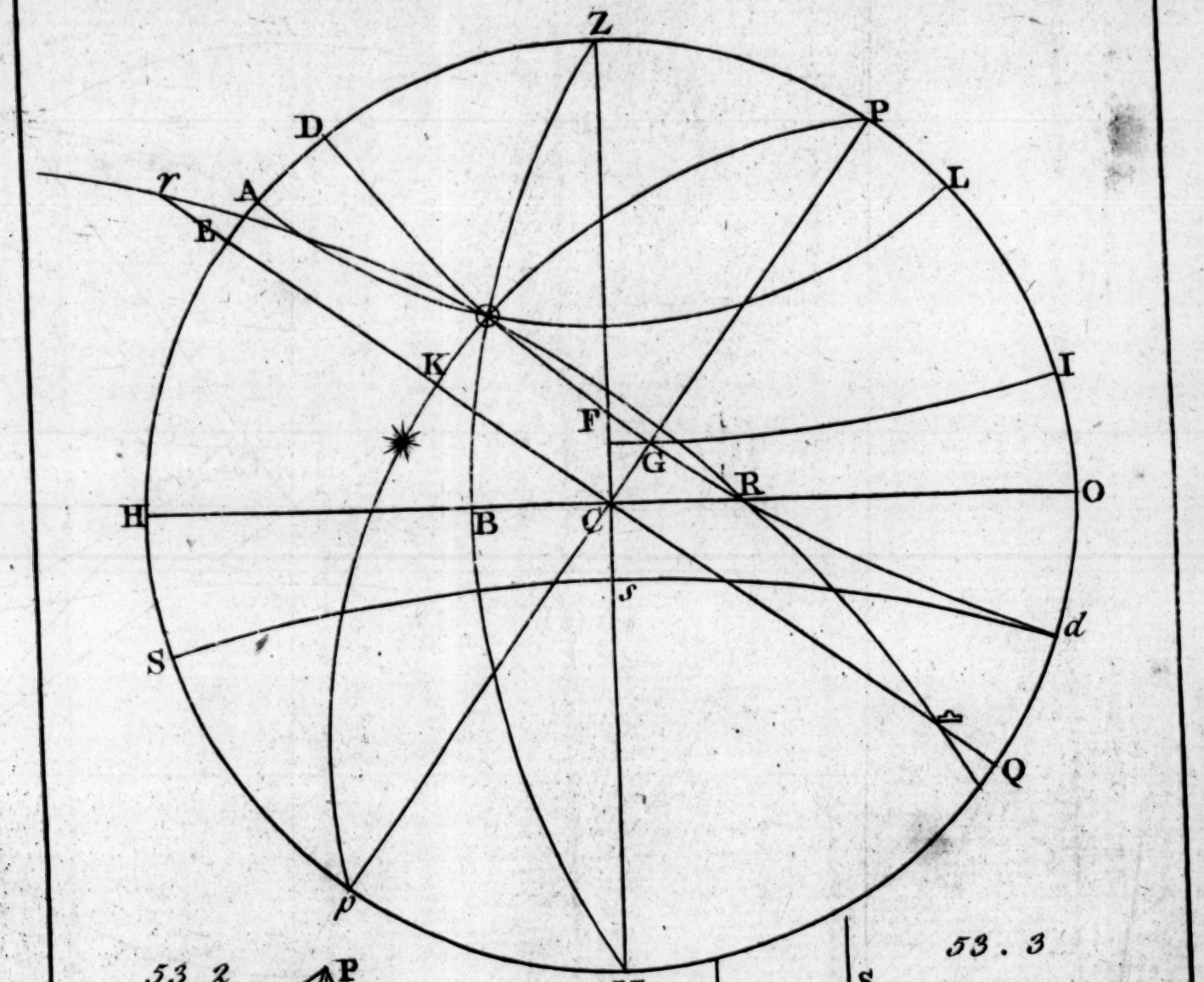


Projection

53.



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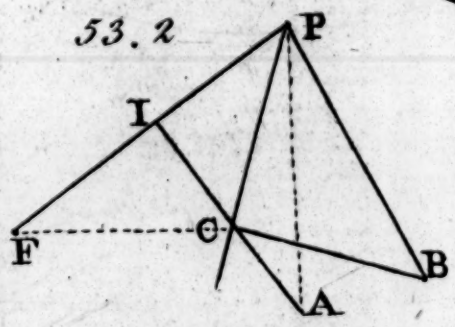

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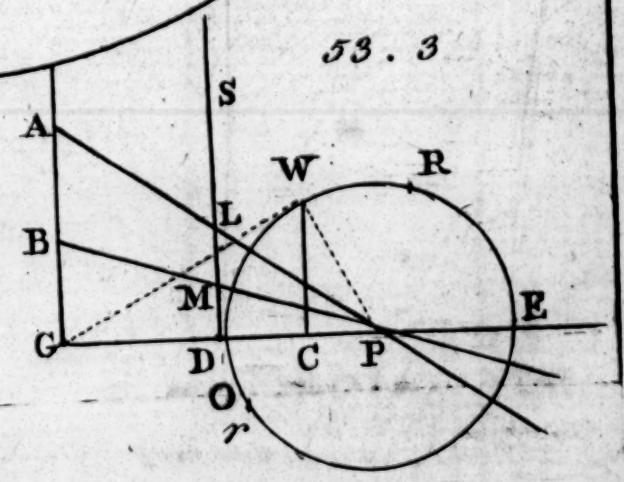
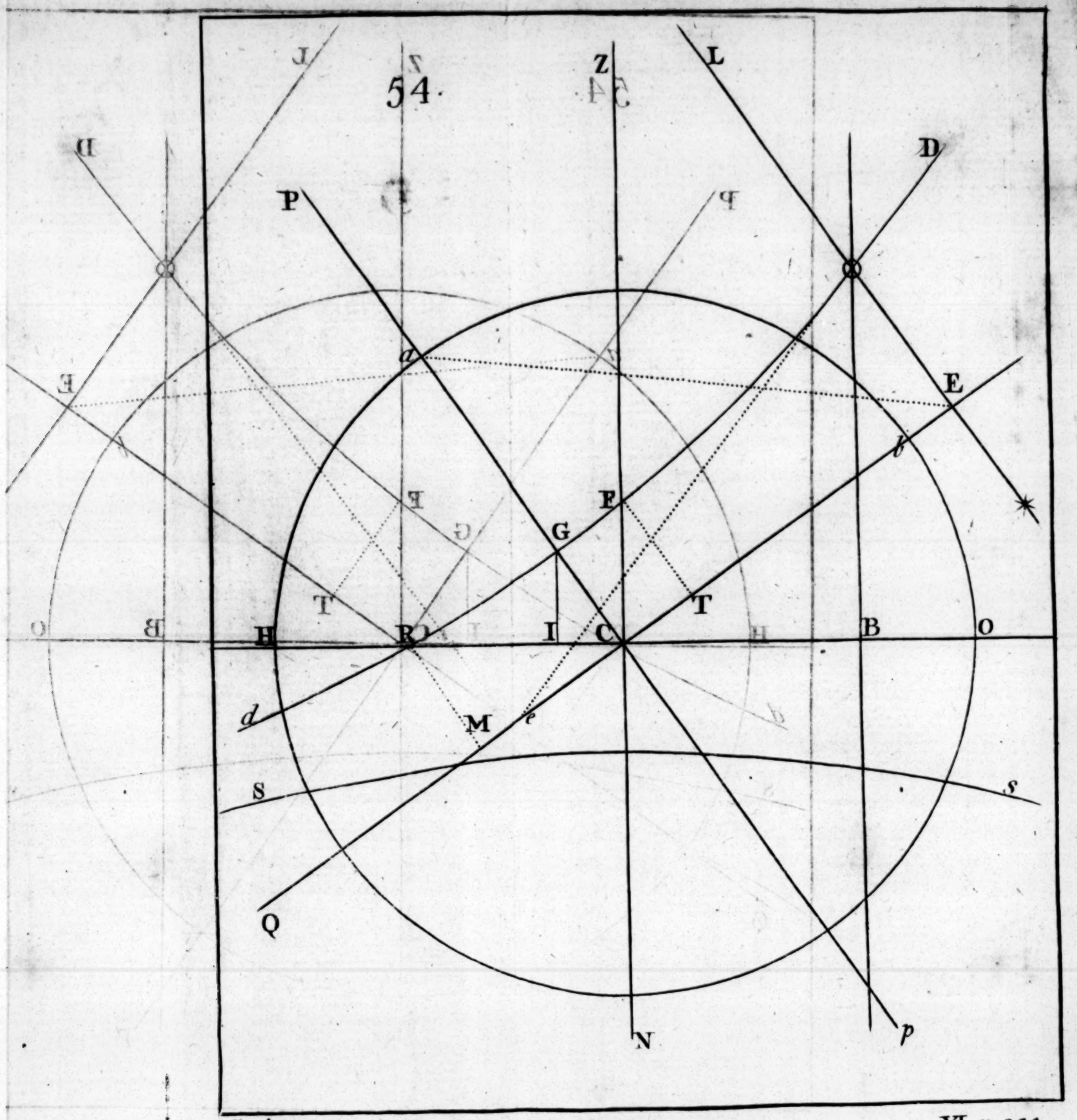


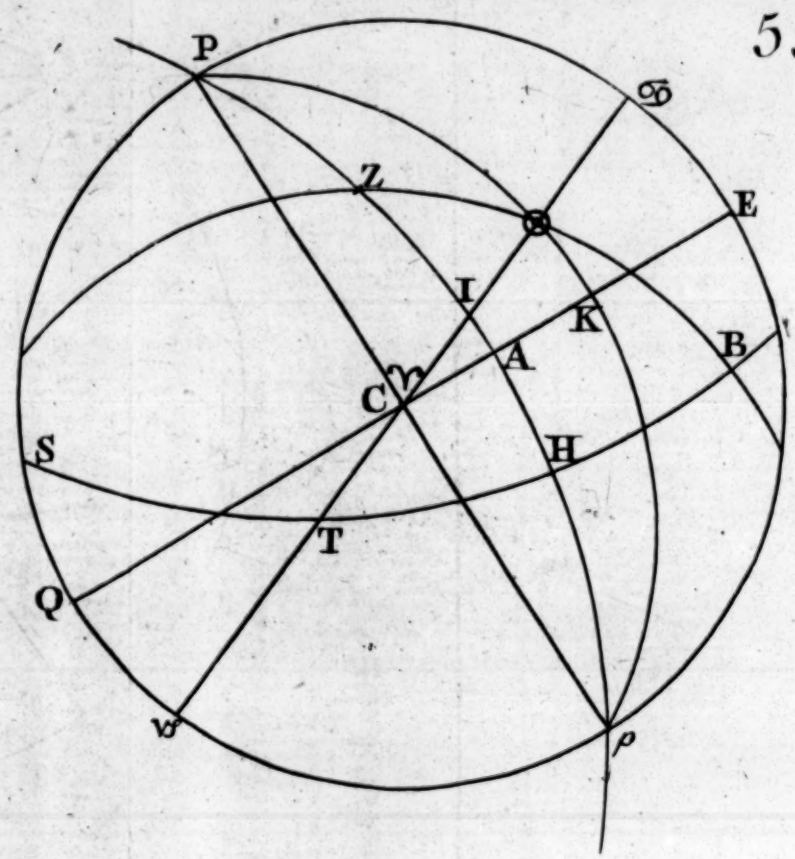
Projection. X. P. 214



Projection.

XI. *p* 274

55.



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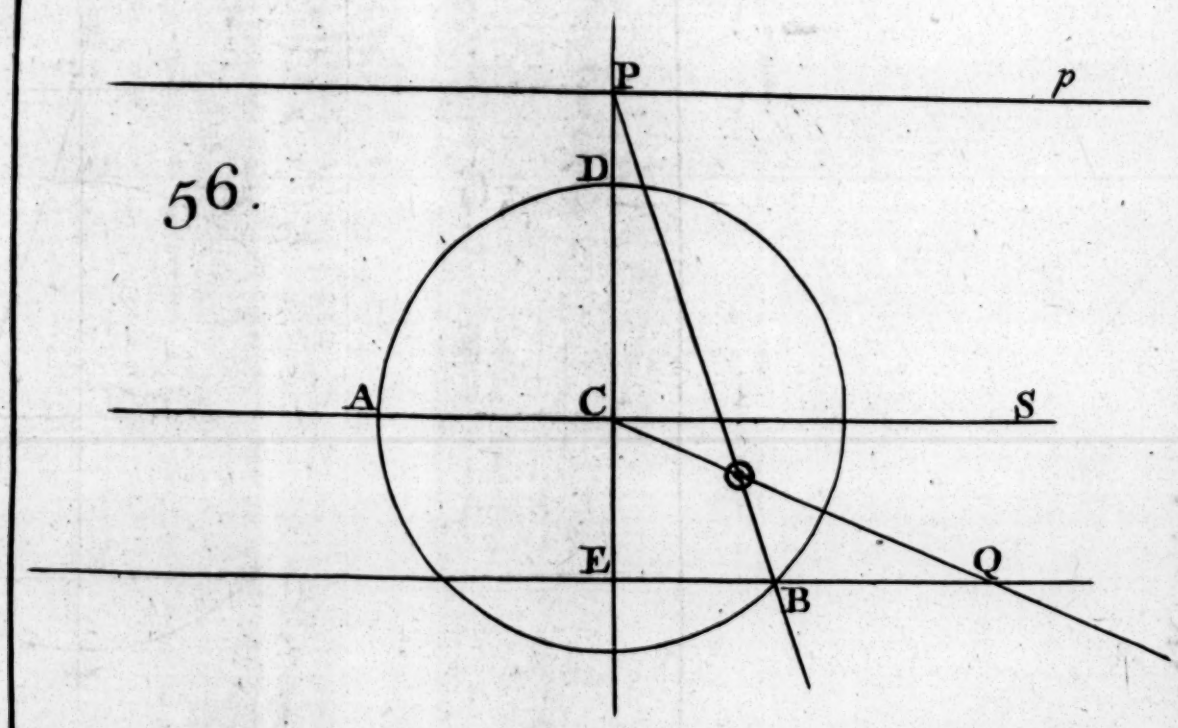
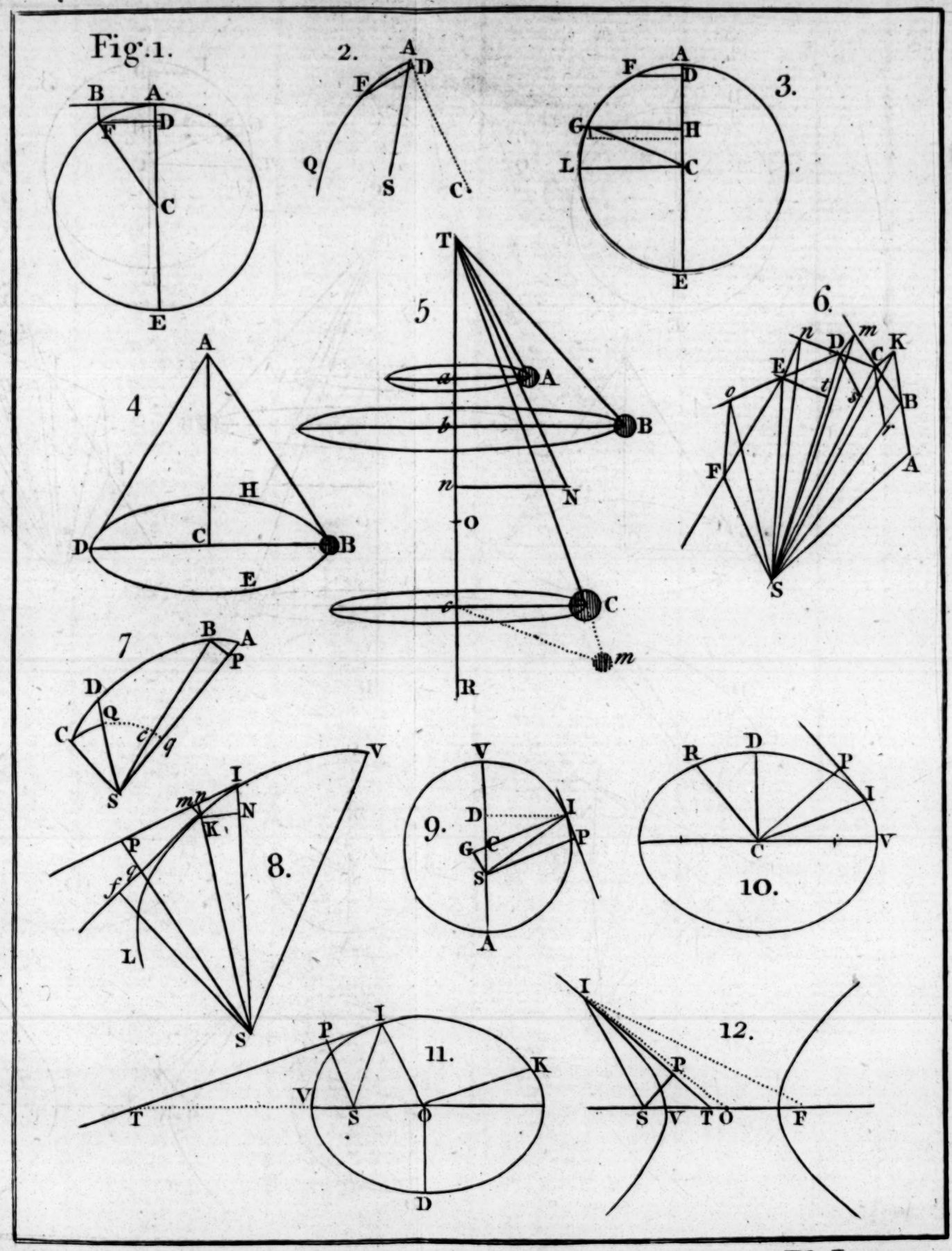


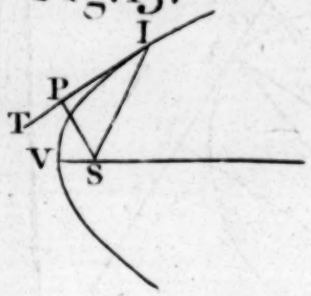
Projection.

XII. *p.* 216



Cent. Forces. Pl. I. *pa.* 246

Fig. 13.



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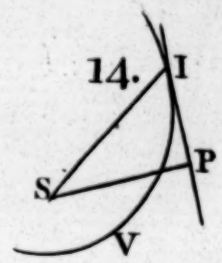

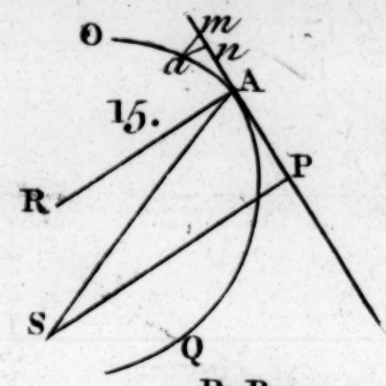

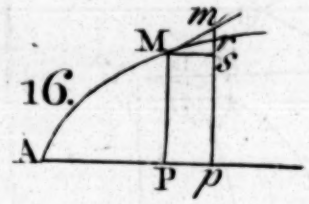

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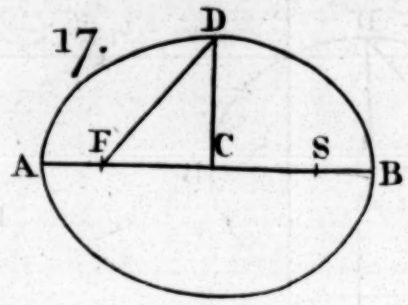
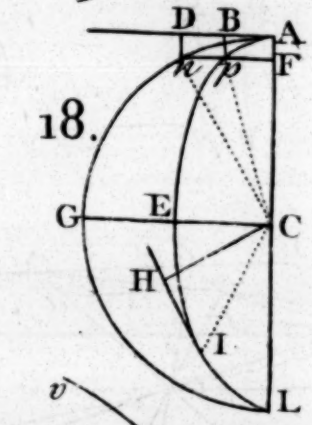
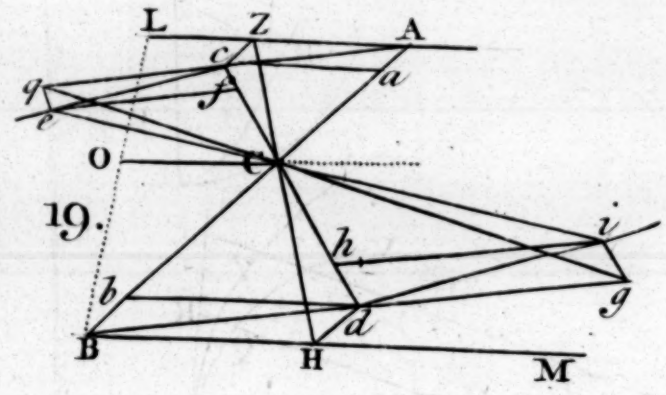
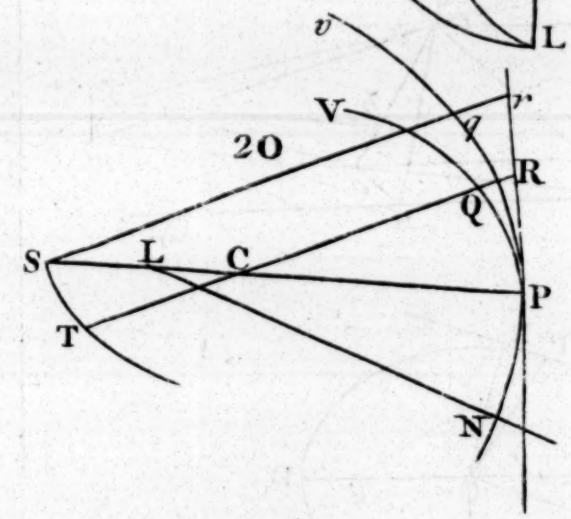
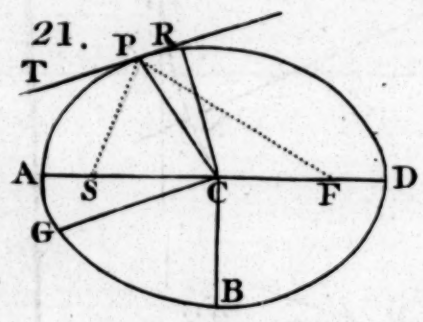
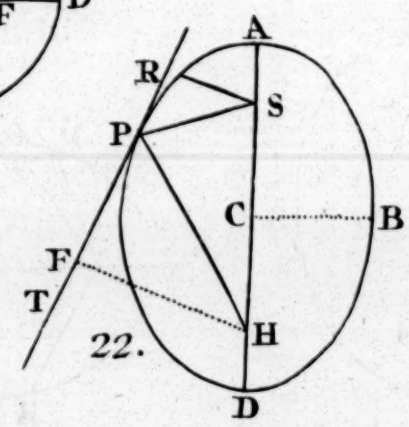


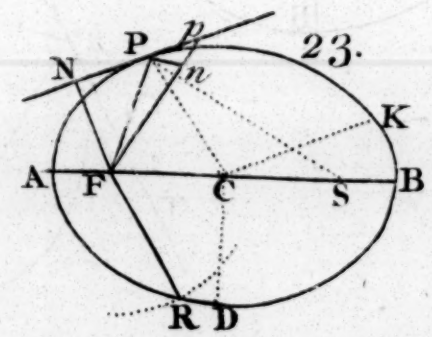








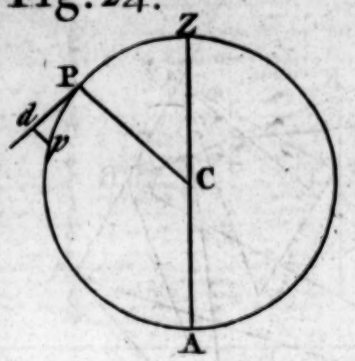


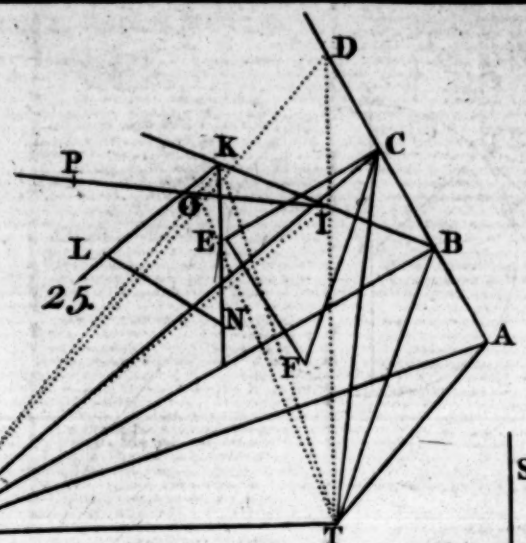


Cent. Forces.

Pl. II. *pa* 266

Fig. 24.





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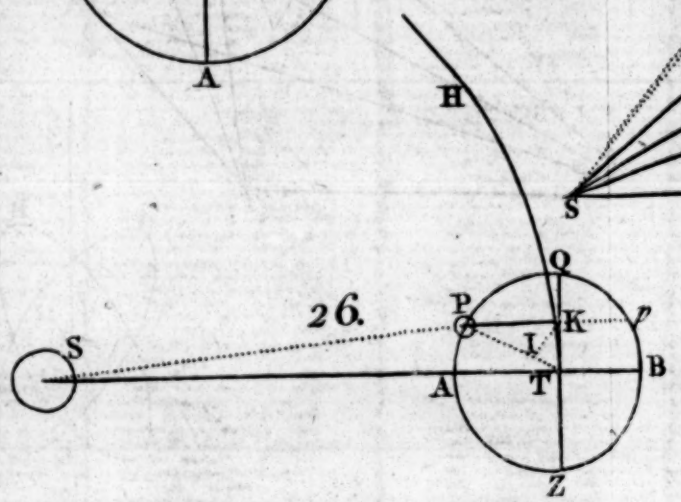
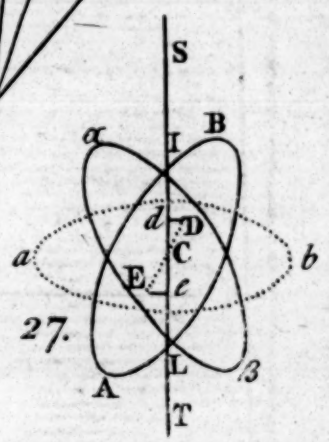
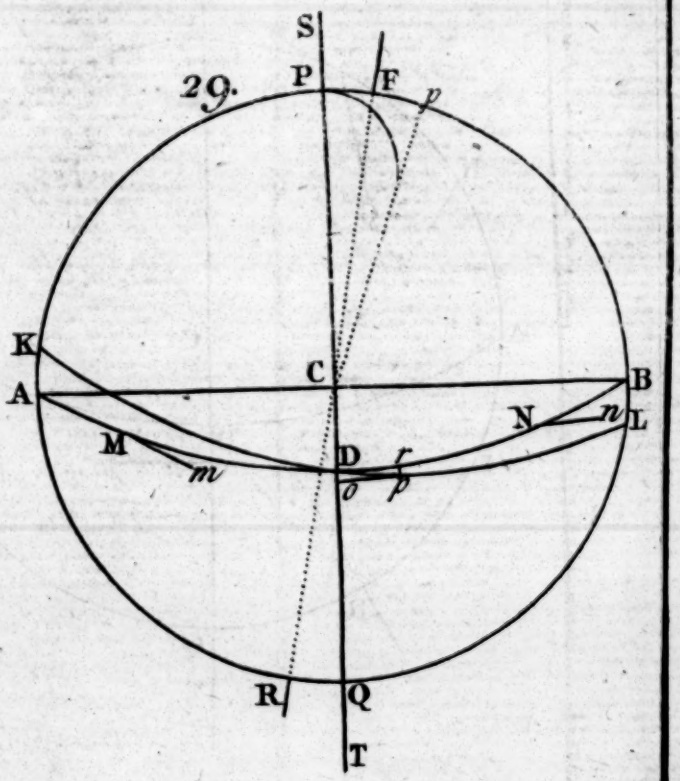


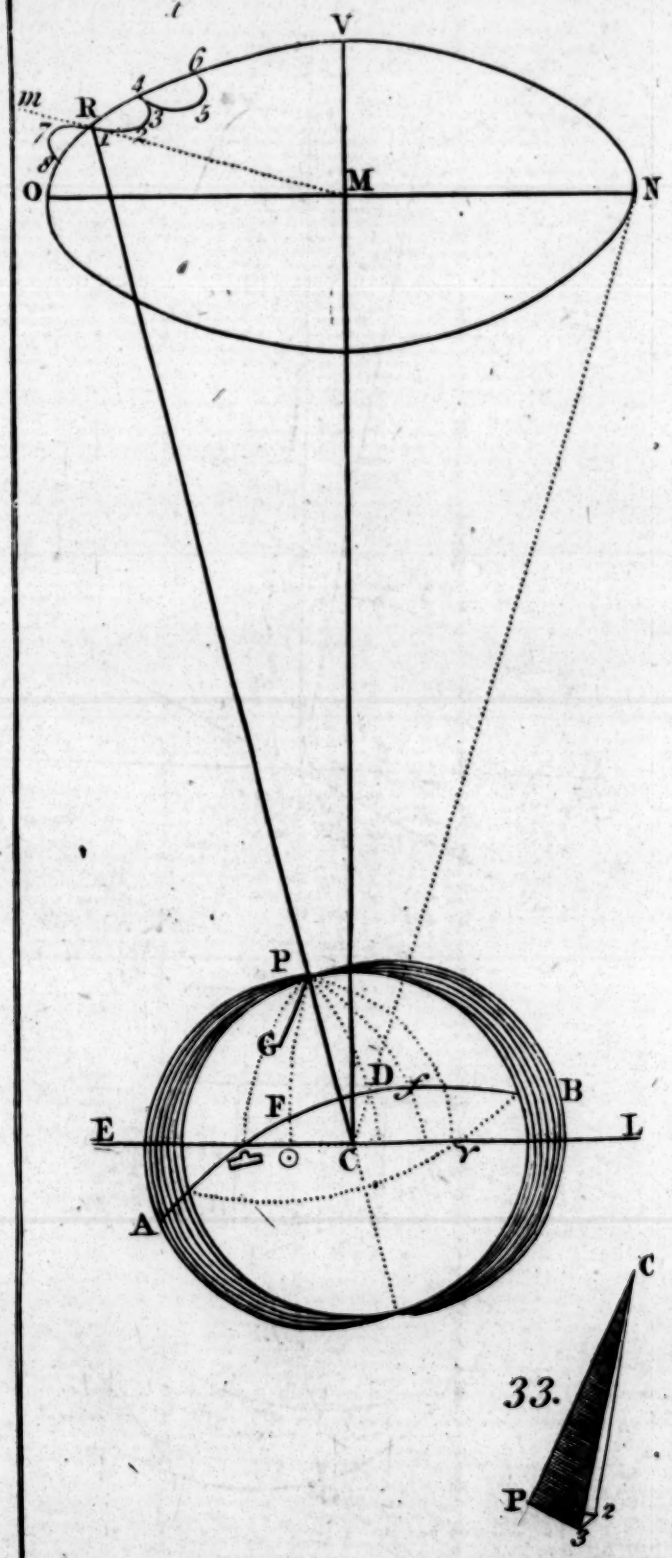


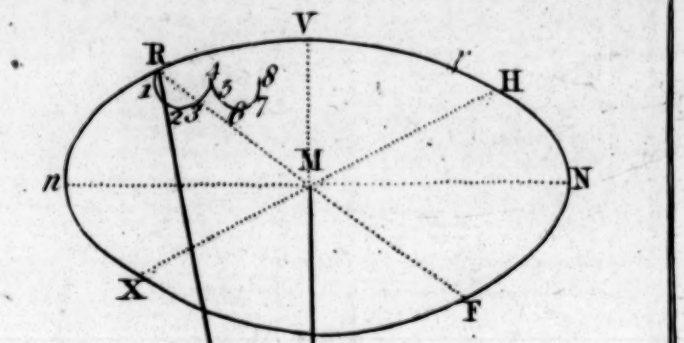


Cent. Forces

Pl. III. pa. 280

Fig. 30.





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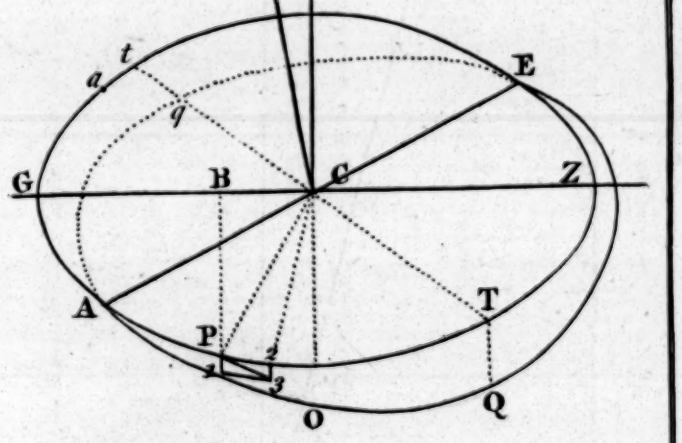

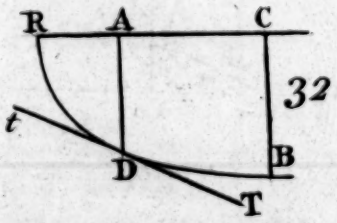

32

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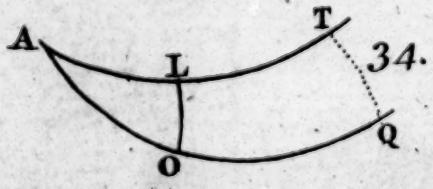
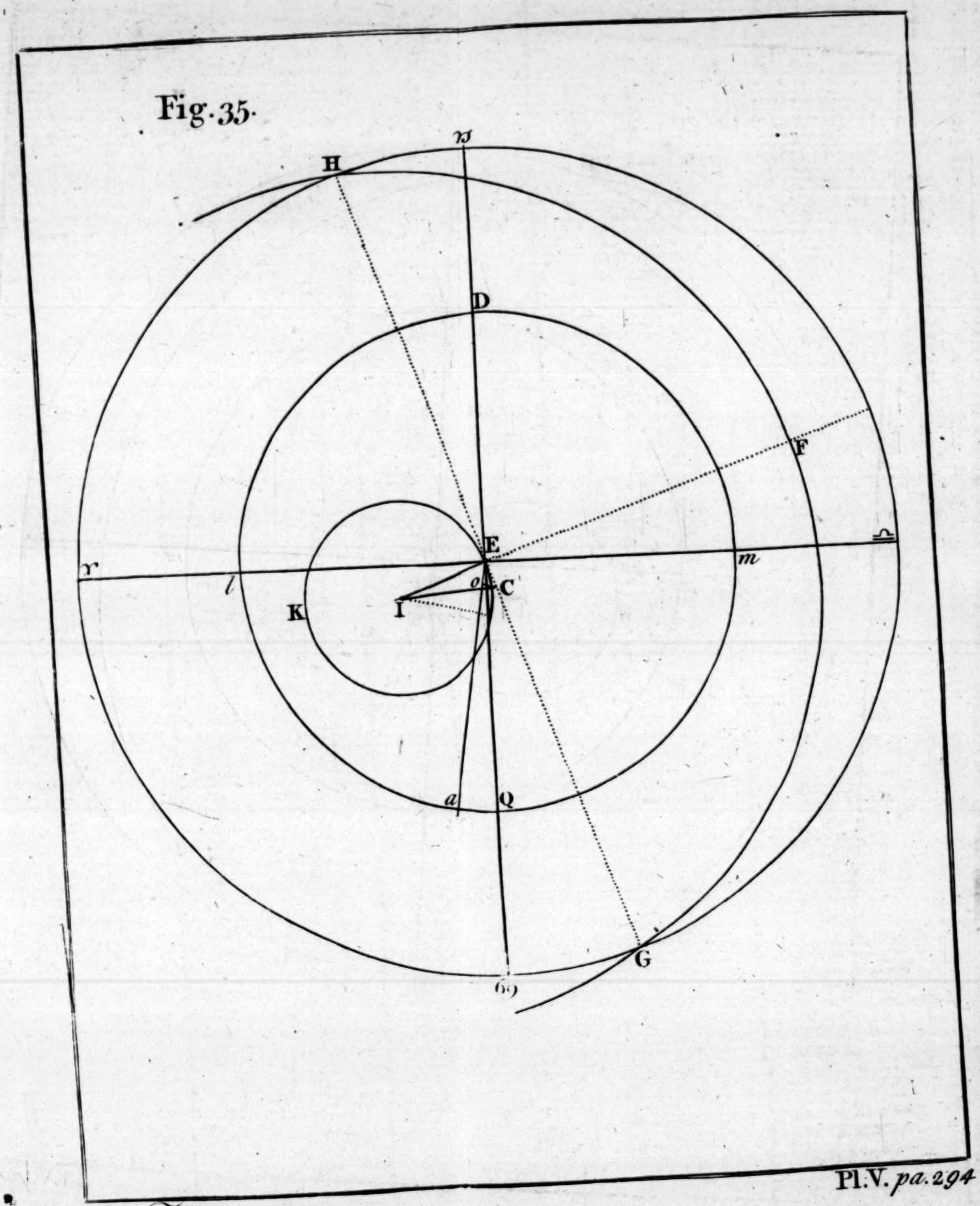


34.

Cent. Forces

Pl. IV. pa 290

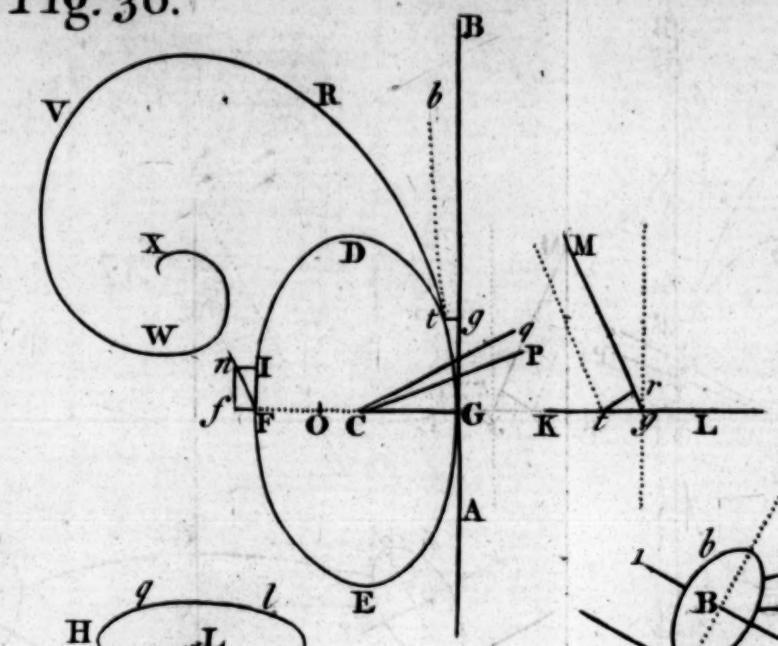
Fig. 35.



Pl. V. *pa.* 294

Cent. Forces

Fig. 36.



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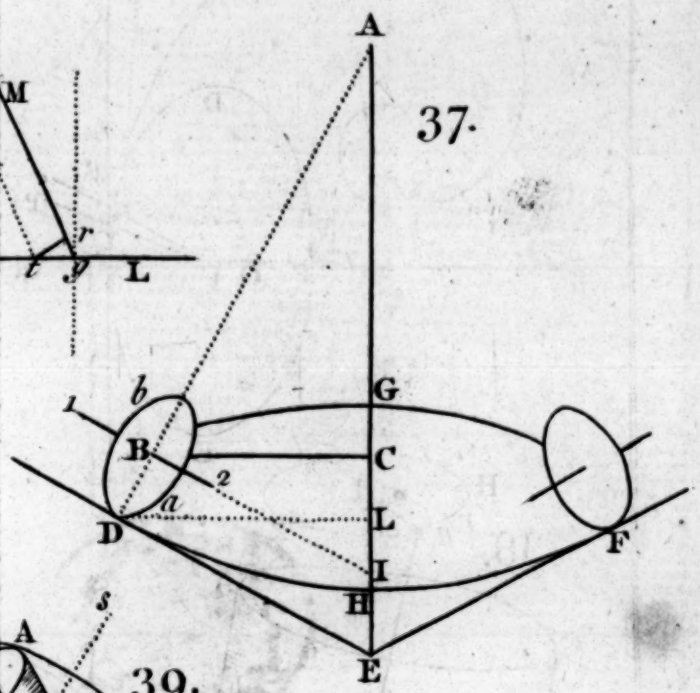

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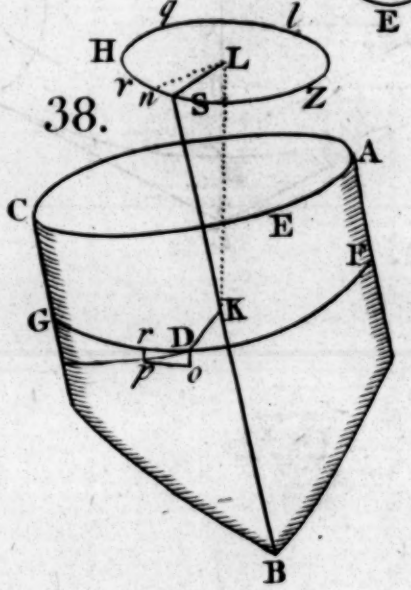

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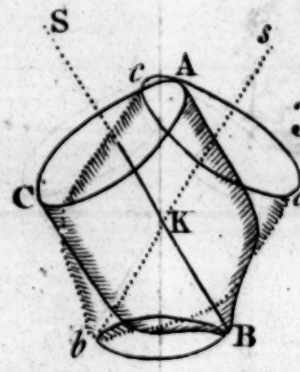

G.R. III.